Two-Degree-of-Freedom Systems

4.1 Introduction

In this chapter we consider methods for writing and solving the equations of motion for some simple two-degree-of-freedom systems. The systems to be studied will all be undamped.

We will begin by examining methods for writing and solving the equations of motion for a free vibration. Knowledge of the characteristics of a free vibration is essential to the consideration of forced vibrations. Next we will deal with the problem of transforming the equations of motion to a more useful set of coordinates. The basic method for solving forced vibration problems involves the transformation to a set of coordinates which uncouples the equations of motion. Some simple forced vibration problems will be examined.

Although matrix methods are not really needed in handling the algebra involved with two-degree-of-freedom systems, we will write some equations in matrix form. Further, we will perform some simple algebraic operations using matrix algebra. Refer to Appendix A as needed for a basic discussion of matrix algebra.

4.2 Equations of Motion for Free Vibrations

Consider the simple mechanical system illustrated by Fig. 4-1. The two rigid bodies, each of mass m, are restrained by two massless springs, each of stiffness k. The constraints permit translation of the bodies in one direction as shown. Let us describe the motion of the system in terms of the translations x_1 and x_2, each measured from the position of static equilibrium. If we disturb the system from the position of static equilibrium, the bodies will experience elastic forces as indicated in the free-body diagrams of Fig. 4-1.

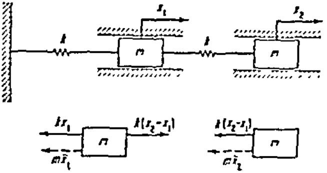

Fig. 4-1 *Simple two-degree-of-freedom system*

Making use of the law of motion for a rigid body in translation, we can write the equations of motion as

$$m\ddot{x}_1 = -2kx_1 + kx_2$$
$$m\ddot{x}_2 = kx_1 - kx_2$$

or

$$\begin{aligned} m\ddot{x}_1 + 2kx_1 - kx_2 &= 0 \\ m\ddot{x}_2 - kx_1 + kx_2 &= 0 \end{aligned} \tag{4.1}$$

Note that the equations of motion are coupled by terms involving the displacements x_1 and x_2. This type of coupling is referred to as static or elastic coupling. The equations of motion for a system may also be coupled by terms involving accelerations. Coupling of this type is called dynamic or inertia coupling. The coupling terms $-kx_2$ and $-kx_1$ have the identical coefficient $-k$. We can say that the elastic coupling is symmetric.

In matrix form, the equations of motion are given by

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{4.2}$$

Elastic coupling is indicated by the off-diagonal terms in the stiffness matrix. The symmetry of the elastic coupling is reflected in the symmetry of the stiffness matrix. Being a diagonal matrix, the inertia matrix is also symmetric.

EXAMPLE 4.1

The double pendulum shown in Fig. 4-2 consists of two particles of mass m and two massless inextensible strings of length l. Let us consider motions of the particles in a common vertical plane. If the strings remain taut during the motion, the system has two degrees of freedom. We can describe the motion of the system by the two angles θ_1 and θ_2 as shown. For the system in motion, the accelerations of the particles in terms of radial and transverse

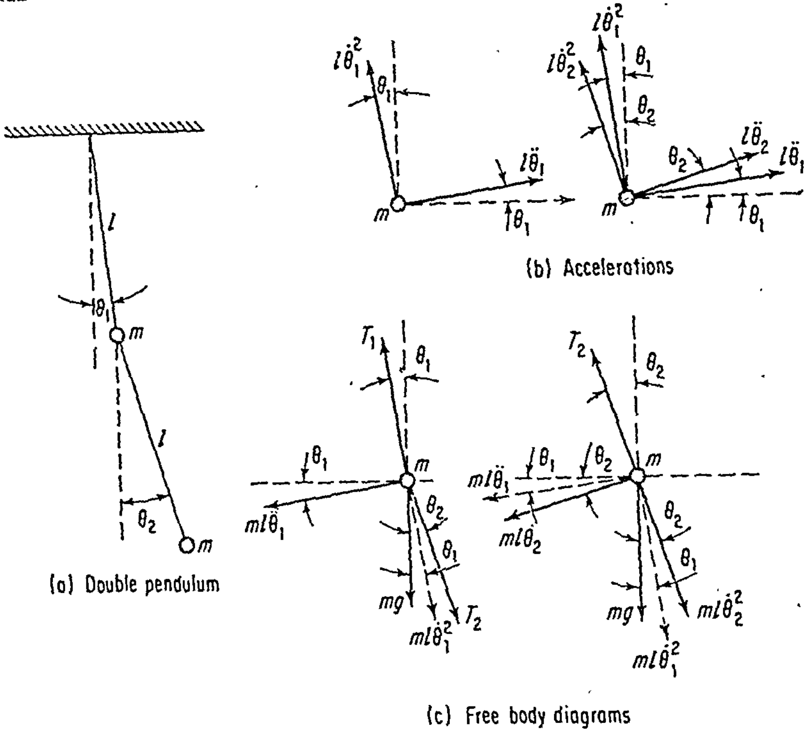

Fig. 4-2 *Double pendulum.*

components are indicated in Fig. 4-2. The acceleration of the lower particle is equal to the sum of the acceleration of the upper particle and the acceleration of the lower particle relative to the upper particle. Making use of D'Alembert's principle, the inertia and gravity forces acting on the particles are given by the free-body diagrams of Fig. 4-2. The string tensions are represented by T_1 and T_2. Dynamic equilibrium of the upper particle is assured if we require the sums of the forces parallel to and normal to the upper string to be zero, leading to

$$\begin{aligned}\sum F_r &= ml\dot{\theta}_1^2 + mg\cos\theta_1 - T_1 + T_2\cos(\theta_2 - \theta_1) = 0\\ \sum F_\perp &= -ml\ddot{\theta}_1 - mg\sin\theta_1 + T_2\sin(\theta_2 - \theta_1) = 0\end{aligned}$$

Let us require the sums of the forces on the lower particle parallel to and normal to the lower string to be zero. Then

$$\begin{aligned}\sum F_r &= ml\dot{\theta}_1^2\cos(\theta_2 - \theta_1) + ml\dot{\theta}_2^2 - ml\ddot{\theta}_1\sin(\theta_2 - \theta_1)\\ &\qquad + mg\cos\theta_2 - T_2 = 0\\ \sum F_\perp &= -ml\ddot{\theta}_1\cos(\theta_2 - \theta_1) - ml\ddot{\theta}_2 - ml\dot{\theta}_1^2\sin(\theta_2 - \theta_1)\\ &\qquad - mg\sin\theta_2 = 0\end{aligned}$$

These equations are nonlinear and we will not attempt to solve them. If we limit the motion to small amplitudes of θ_1 and θ_2, we can introduce the approximations

$$\sin\theta_1 \approx \theta_1$$
$$\sin\theta_2 \approx \theta_2$$
$$\sin(\theta_2 - \theta_1) \approx \theta_2 - \theta_1$$
$$\cos\theta_1 = \cos\theta_2 = \cos(\theta_2 - \theta_1) \approx 1$$

We can reason that the velocities $\dot{\theta}_1$ and $\dot{\theta}_2$ and the accelerations $\ddot{\theta}_1$ and $\ddot{\theta}_2$ will be small and that products of small quantities can be ignored. Introducing the approximations and keeping only the larger terms, we can simplify the equations of motion to

$$mg - T_1 + T_2 = 0$$
$$-ml\ddot{\theta}_1 - mg\theta_1 + T_2(\theta_2 - \theta_1) = 0$$
$$mg - T_2 = 0$$
$$-ml\ddot{\theta}_1 - ml\ddot{\theta}_2 - mg\theta_2 = 0$$

The string tensions are

$$T_1 = 2mg$$
$$T_2 = mg$$

and the linearized equations of motion for the double pendulum are

$$-ml\ddot{\theta}_1 - 2mg\theta_1 + mg\theta_2 = 0$$
$$-ml\ddot{\theta}_1 - ml\ddot{\theta}_2 - mg\theta_2 = 0$$

Note that the coupling of the equations is confused. The first equation is coupled to the second by the gravity term $mg\theta_2$ while the second equation is coupled to the first by the inertia term $-ml\ddot{\theta}_1$. As written, the coupling of the equations is not symmetric. In Example 4.4 we will discover that the equations of motion can be written in a symmetric form. This can be accomplished by replacing the first equation by the sum of the two equations.

4.3 Influence Coefficients

In deriving the equations of motion for the system of Fig. 4-1, the force-displacement relationships were described in terms of the stiffnesses of the individual springs. A more convenient method for describing the force-displacement relationships for a complete system involves the use of a set of influence coefficients. Let us consider the displacement conditions of the system shown in Fig. 4-3. The forces on the bodies required to produce displacement x_1 are given by $F_1 = k_{11}x_1$ and $F_2 = k_{21}x_1$. Similarly, the forces required to produce displacement x_2 are given by $F_1 = k_{12}x_2$ and $F_2 = k_{22}x_2$. The quantities k are referred to as the *stiffness influence coefficients* of the system. Note that the influence coefficients have been defined in

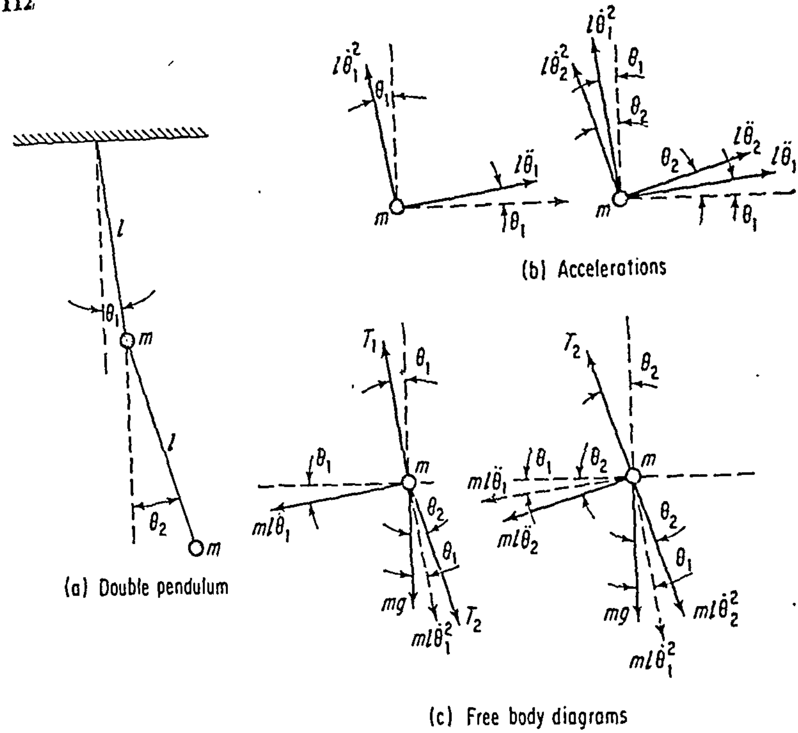

Fig. 4-2 *Double pendulum.*

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$$\begin{aligned}\sum F_{\parallel} &= ml\dot{\theta}_1^2 + mg\cos\theta_1 - T_1 + T_2\cos(\theta_2 - \theta_1) = 0\\ \sum F_{\perp} &= -ml\ddot{\theta}_1 - mg\sin\theta_1 + T_2\sin(\theta_2 - \theta_1) = 0\end{aligned}$$

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$$\begin{aligned}\sum F_{\parallel} &= ml\dot{\theta}_1^2\cos(\theta_2 - \theta_1) + ml\dot{\theta}_2^2 - ml\ddot{\theta}_1\sin(\theta_2 - \theta_1)\\ &\qquad + mg\cos\theta_2 - T_2 = 0\\ \sum F_{\perp} &= -ml\ddot{\theta}_1\cos(\theta_2 - \theta_1) - ml\ddot{\theta}_2 - ml\dot{\theta}_1^2\sin(\theta_2 - \theta_1)\\ &\qquad - mg\sin\theta_2 = 0\end{aligned}$$

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$$\begin{aligned} \sin\theta_1 &\approx \theta_1 \\ \sin\theta_2 &\approx \theta_2 \\ \sin(\theta_2 - \theta_1) &\approx \theta_2 - \theta_1 \\ \cos\theta_1 = \cos\theta_2 = \cos(\theta_2 - \theta_1) &\approx 1 \end{aligned}$$

We can reason that the velocities $\dot{\theta}_1$ and $\dot{\theta}_2$ and the accelerations $\ddot{\theta}_1$ and $\ddot{\theta}_2$ will be small and that products of small quantities can be ignored. Introducing the approximations and keeping only the larger terms, we can simplify the equations of motion to

$$\begin{aligned} mg - T_1 + T_2 &= 0 \\ -ml\ddot{\theta}_1 - mg\theta_1 + T_2(\theta_2 - \theta_1) &= 0 \\ mg - T_2 &= 0 \\ -ml\ddot{\theta}_1 - ml\ddot{\theta}_2 - mg\theta_2 &= 0 \end{aligned}$$

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and the linearized equations of motion for the double pendulum are

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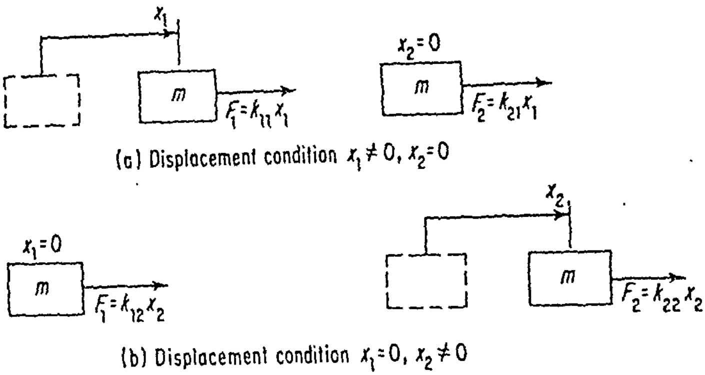


Fig. 4-3 *Definition of the stiffness influence coefficients.*

terms of external forces applied to the bodies. The elastic forces acting on the bodies will be the negatives of the applied forces. It is not difficult to show that the stiffness influence coefficients for the simple system of Fig. 4-1 are given by

$$\begin{aligned} k_{11} &= 2k \\ k_{12} &= k_{21} = -k \\ k_{22} &= k \end{aligned} \tag{4.3}$$

The fact that $k_{12} = k_{21}$ is referred to as the symmetry property. An arbitrary displacement of the system, x_1, x_2 is represented by superposition of the two

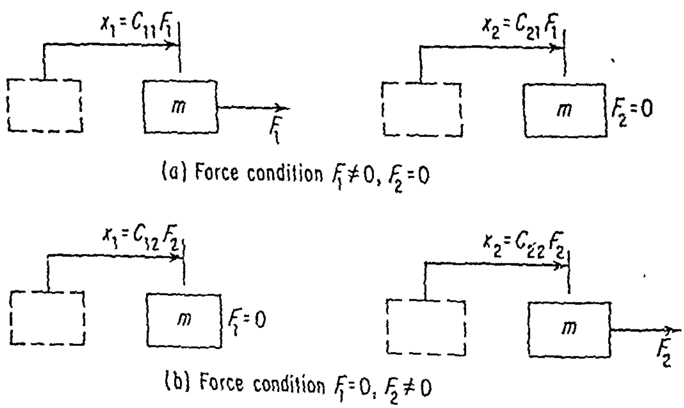


Fig. 4-4 *Definition of the flexibility influence coefficients.*

displacement conditions of Fig. 4-3. The forces required on the bodies to give the system an arbitrary displacement are

$$\begin{aligned} F_1 &= k_{11}x_1 + k_{12}x_2 = 2kx_1 - kx_2 \\ F_2 &= k_{21}x_1 + k_{22}x_2 = -kx_1 + kx_2 \end{aligned} \tag{4 4}$$

In matrix form, we can write

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \tag{4.5}$$

In Fig. 4-4, the system is shown under two different conditions of loading. The displacements of the bodies resulting from the force F_1 are given by $x_1 = C_{11}F_1$ and $x_2 = C_{21}F_1$ Similarly the displacements of the bodies *resulting from the force F_2 are given by $x_1 = C_{12}F_2$ and $x_2 = C_{22}F_2$ The* quantities C are referred to as the flexibility influence coefficients of the system. From the definition, the flexibility influence coefficients for the simple system of Fig. 4-1 are

$$\begin{aligned} C_{11} &= C_{12} = C_{21} = \frac{1}{k} \\ C_{22} &= \frac{2}{k} \end{aligned} \tag{4.6}$$

Note the symmetry property, represented by $C_{12} = C_{21}$ Superposition of the two loading conditions of Fig 4-4 represents an arbitrary loading of the system. We can write the displacements resulting from an arbitrary loading as

$$\begin{aligned} x_1 &= C_{11}F_1 + C_{12}F_2 = \frac{1}{k}F_1 + \frac{1}{k}F_2 \\ x_2 &= C_{21}F_1 + C_{22}F_2 = \frac{1}{k}F_1 + \frac{2}{k}F_2 \end{aligned} \tag{4.7}$$

or, in matrix form, by

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} \frac{1}{k} & \frac{1}{k} \\ \frac{1}{k} & \frac{2}{k} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \tag{4 8}$$

The force-displacement relationships for the system of Fig. 4-1 are given by either Eq. (4.4) or (4.7), or in matrix form by either Eq. (4 5) or (4.8). Evi*dently the stiffness and flexibility influence coefficients are related.* If we know either set of coefficients, we can determine the other set making use of the given relationships.

For a free vibration of the system of Fig. 4-1, the forces other than elastic forces which are acting on the two bodies are the inertia forces

$$\begin{aligned} F_1 &= -m\ddot{x}_1 \\ F_2 &= -m\ddot{x}_2 \end{aligned} \tag{4.9}$$

or

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = -\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} \tag{4.10}$$

Substitution of Eqs. (4.9) into Eqs. (4.4) leads to the equations of motion

$$\begin{aligned} -m\ddot{x}_1 &= 2kx_1 - kx_2 \\ -m\ddot{x}_2 &= -kx_1 + kx_2 \end{aligned} \tag{4.11}$$

which compare with Eqs. (4.1). Similarly, the matrix form of the equations of motion, Eqs. (4.2), results from the combination of Eqs. (4.5) and (4.10). Substitution of Eqs. (4.9) into Eqs. (4.7) leads to an alternate form for the equations of motion, given by

$$\begin{aligned} x_1 &= -\frac{m}{k}\ddot{x}_1 - \frac{m}{k}\ddot{x}_2 \\ x_2 &= -\frac{m}{k}\ddot{x}_1 - \frac{2m}{k}\ddot{x}_2 \end{aligned} \tag{4.12}$$

From Eqs. (4.8) and (4.10), we can write

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = -\begin{bmatrix} \frac{1}{k} & \frac{1}{k} \\ \frac{1}{k} & \frac{2}{k} \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}$$

which, after multiplication of the square matrices, leads to

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = -\begin{bmatrix} \frac{m}{k} & \frac{m}{k} \\ \frac{m}{k} & \frac{2m}{k} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} \tag{4.13}$$

This is the matrix form of Eqs. (4.12).

EXAMPLE *4.2*

A system is made up of two identical rigid disks restrained by a massless elastic shaft as shown in Fig. 4-5. Each of the disks has an axial mass moment of inertia I. Each of the shaft segments is characterized by the torsional stiffness k. We can describe a motion of axial rotation of the system by the two angles θ_1 and θ_2. Let us write the force-displacement relationships in terms of

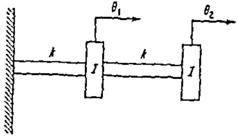


Fig. 4-5

the flexibility influence coefficients. From the definition, it can easily be shown that

$$C_{11} = C_{12} = C_{21} = \frac{1}{k}$$
$$C_{22} = \frac{2}{k}$$

The torques other than elastic torques acting on the disks in a free vibration are the inertia torques

$$T_1 = -I\ddot{\theta}_1$$
$$T_2 = -I\ddot{\theta}_2$$

Then the rotations of the disks resulting from the inertia loading are given by

$$\theta_1 = C_{11}T_1 + C_{12}T_2 = -\frac{I}{k}\ddot{\theta}_1 - \frac{I}{k}\ddot{\theta}_2$$
$$\theta_2 = C_{21}T_1 + C_{22}T_2 = -\frac{I}{k}\ddot{\theta}_1 - \frac{2I}{k}\ddot{\theta}_2$$

These are the equations of motion for the system. In matrix form

$$\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = -\begin{bmatrix}\frac{I}{k} & \frac{I}{k}\\ \frac{I}{k} & \frac{2I}{k}\end{bmatrix}\begin{Bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{Bmatrix}$$

EXAMPLE 4.3

Suppose one of the shaft segments of the system of Fig. 4-5 has been removed, resulting in the system of Fig 4-6. It has been our practice to place the origin of coordinates at the position of static equilibrium. However, for this system there is not a unique position of static equilibrium. If we place the origins of θ_1 and θ_2 at a position of static equilibrium, any position of the

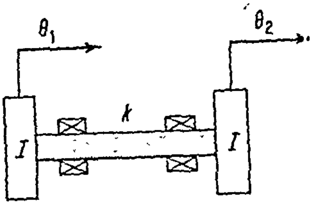


Fig. 4-6

system with $\theta_1 = \theta_2$ is also a position of static equilibrium. The situation just described will occur whenever the constraints are not sufficient to prevent rigid body motion of the system.

If we apply a torque to either disk, the system will experience rotation without limit and will never reach a condition of static equilibrium. It is evident that the concept of flexibility influence coefficients cannot be used in this example. Let us then describe the force-displacement relationships in terms of the stiffness influence coefficients. From the definition, we can show that

$$k_{11} = k_{22} = k$$
$$k_{12} = k_{21} = -k$$

For an arbitrary rotation of the disks

$$\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$$

As in Example 4.2, the inertia torques are

$$\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = -\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix}$$

Then the equations of motion for a free vibration of the system are

$$-\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$$

4.4 The Work and Energy Approach

Let us derive the equations of motion for the system of Fig. 4-1 using the principle of virtual displacements. For the system in motion, the elastic and inertia forces acting on the bodies at an instant are shown in the free-body diagrams of Fig. 4-1. A virtual displacement of the system is represented by the displacements δx_1 and δx_2. The work done by the elastic and inertia

forces in a virtual displacement from the position of dynamic equilibrium is given by

$$\delta W = (-m\ddot{x}_1 - 2kx_1 + kx_2)\,\delta x_1 + (-m\ddot{x}_2 + kx_1 - kx_2)\,\delta x_2 = 0 \tag{4.14}$$

The displacements δx_1 and δx_2 being arbitrary, their coefficients must be zero, leading to the equations of motion

$$\begin{aligned} \sum F_1 &= -m\ddot{x}_1 - 2kx_1 + kx_2 = 0 \\ \sum F_2 &= -m\ddot{x}_2 + kx_1 - kx_2 = 0 \end{aligned} \tag{4.15}$$

The generalized inertia forces associated with x_1 and x_2 are

$$\begin{aligned} F_{1,in} &= -m\ddot{x}_1 \\ F_{2,in} &= -m\ddot{x}_2 \end{aligned} \tag{4.16}$$

Similarly, the generalized elastic forces associated with x_1 and x_2 are

$$\begin{aligned} F_{1,el} &= -2kx_1 + kx_2 \\ F_{2,el} &= kx_1 - kx_2 \end{aligned} \tag{4.17}$$

Each of the equations of motion is a statement that the generalized forces associated with a coordinate are in dynamic equilibrium.

The expression of the principle of virtual displacements, Eq. (4.14), can be written in matrix form as

$$\delta W = \lfloor \delta x_1 \;\; \delta x_2 \rfloor \begin{Bmatrix} \sum F_1 \\ \sum F_2 \end{Bmatrix} = 0 \tag{4.18}$$

in which

$$\begin{Bmatrix} \sum F_1 \\ \sum F_2 \end{Bmatrix} = -\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} - \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \tag{4.19}$$

Since the virtual displacements δx_1 and δx_2 are arbitrary, we can conclude from Eq. (4.18) that

$$\begin{Bmatrix} \sum F_1 \\ \sum F_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{4.20}$$

The equations of motion are given by Eqs. (4.19) and (4.20). We can identify the generalized inertia and elastic forces as

$$\begin{aligned} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}_{in} &= -\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} \\ \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}_{el} &= -\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \end{aligned} \tag{4.21}$$

These results agree with those of Eqs. (4.16) and (4.17).

The generalized forces can be obtained conveniently from the expressions for the kinetic and potential energy. We can write

$$T = \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}m\dot{x}_2^2$$
$$U = \tfrac{1}{2}kx_1^2 + \tfrac{1}{2}k(x_2 - x_1)^2 \tag{4.22}$$

Then the generalized inertia forces are

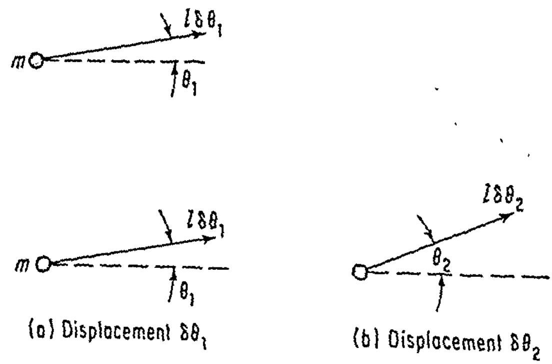
$$F_{1,in} = -\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = -m\ddot{x}_1$$
$$F_{2,in} = -\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = -m\ddot{x}_2 \tag{4.23}$$

Similarly the generalized elastic forces are

$$F_{1,el} = -\frac{\partial U}{\partial x_1} = -2kx_1 + kx_2$$
$$F_{2,el} = -\frac{\partial U}{\partial x_2} = kx_1 - kx_2 \tag{4.24}$$

EXAMPLE 4.4

We can write the equations of motion for the double pendulum of Example 4.1 in a more systematic way by using the principle of virtual displacements. For a free vibration, the forces acting on the particles are shown in the free-body diagram of Fig. 4-2. A virtual displacement is represented by the angular displacements $\delta\theta_1$ and $\delta\theta_2$. As shown in Fig. 4-7, the displacement $\delta\theta_1$ involves equal displacements of both particles while the displacement



(a) Displacement $\delta\theta_1$ (b) Displacement $\delta\theta_2$

Fig. 4-7 *Virtual displacement of double pendulum.*

$\delta\theta_2$ involves only a displacement of the lower particle. The work done in the virtual displacement is given by

$$\begin{aligned}\delta W &= [-2ml\ddot{\theta}_1 - ml\ddot{\theta}_2 \cos(\theta_2 - \theta_1) + ml\dot{\theta}_2^2 \sin(\theta_2 - \theta_1) \\ &\qquad - 2mg \sin\theta_1]l\,\delta\theta_1 \\ &\quad + [-ml\ddot{\theta}_1 \cos(\theta_2 - \theta_1) - ml\ddot{\theta}_2 - ml\dot{\theta}_1^2 \sin(\theta_2 - \theta_1) \\ &\qquad - mg \sin\theta_2]l\,\delta\theta_2 \\ &= 0\end{aligned}$$

Note that the unknown tensions T_1 and T_2 do not work. It is characteristic of the principle of virtual displacements that forces of constraint such as the string tensions do no work. As a result the forces of constraint will not appear in the equations of motion. Since $\delta\theta_1$ and $\delta\theta_2$ are arbitrary, the equations of motion are

$$\begin{aligned}\sum F_1 &= -2ml^2\ddot{\theta}_1 - ml^2\ddot{\theta}_2 \cos(\theta_2 - \theta_1) + ml^2\dot{\theta}_2^2 \sin(\theta_2 - \theta_1) \\ &\qquad\qquad -2mgl \sin\theta_1 = 0 \\ \sum F_2 &= -ml^2\ddot{\theta}_1 \cos(\theta_2 - \theta_1) - ml^2\ddot{\theta}_2 - ml^2\dot{\theta}_1^2 \sin(\theta_2 - \theta_1) \\ &\qquad\qquad -mgl \sin\theta_2 = 0\end{aligned}$$

The generalized inertia and gravity forces associated with θ_1 and θ_2 are

$$\begin{aligned}F_{1,in} &= -2ml^2\ddot{\theta}_1 - ml^2\ddot{\theta}_2 \cos(\theta_2 - \theta_1) + ml^2\dot{\theta}_2^2 \sin(\theta_2 - \theta_1) \\ F_{2,in} &= -ml^2\ddot{\theta}_1 \cos(\theta_2 - \theta_1) - ml^2\ddot{\theta}_2 - ml^2\dot{\theta}_1^2 \sin(\theta_2 - \theta_1) \\ F_{1,g} &= -2mgl \sin\theta_1 \\ F_{2,g} &= -mgl \sin\theta_2\end{aligned}$$

Since the generalized forces are associated with an angle, they have the dimensions of the moment of a force.

It is usually more convenient to derive the generalized forces from the expressions for the kinetic and potential energies. We can write the energy expressions as

$$\begin{aligned}T &= \tfrac{1}{2}m(l\dot{\theta}_1)^2 + \tfrac{1}{2}m[(l\dot{\theta}_1)^2 + (l\dot{\theta}_2)^2 + 2l^2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_2 - \theta_1)] \\ U &= mgl(1 - \cos\theta_1) + mgl(2 - \cos\theta_1 - \cos\theta_2)\end{aligned}$$

The generalized inertia and gravity forces are given by

$$\begin{aligned}F_{1,in} &= -\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) + \frac{\partial T}{\partial \theta_1} \\ F_{2,in} &= -\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) + \frac{\partial T}{\partial \theta_2} \\ F_{1,g} &= -\frac{\partial U}{\partial \theta_1} \\ F_{2,g} &= -\frac{\partial U}{\partial \theta_2}\end{aligned}$$

As in Example 4.1, the equations of motion are non-linear. If we consider motions of small amplitude in θ_1 and θ_2, we can introduce approximations as was done in Example 4.1. The resulting linearized equations of motion are

$$\sum F_1 = -2ml^2\ddot{\theta}_1 - ml^2\ddot{\theta}_2 - 2mgl\theta_1 = 0$$
$$\sum F_2 = -ml^2\ddot{\theta}_1 - ml^2\ddot{\theta}_2 - mgl\theta_2 = 0$$

The equations of motion are inertially coupled. Since the coupling terms $-ml^2\ddot{\theta}_2$ and $-ml^2\ddot{\theta}_1$ have the identical coefficient $-ml^2$, the coupling is symmetric. Comparing the results with those of Example 4.1, we will prefer the symmetrically coupled equations of motion given here. Note that the equations of motion given here are statements that the sums of the generalized forces associated with θ_1 and θ_2 are separately zero. In matrix form we can write

$$\begin{Bmatrix} \sum F_1 \\ \sum F_2 \end{Bmatrix} = -\begin{bmatrix} 2ml^2 & ml^2 \\ ml^2 & ml^2 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} - \begin{bmatrix} 2mgl & 0 \\ 0 & mgl \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

EXAMPLE 4.5

In the idealized system of Fig. 4-8, two particles of mass m and $2m$ are fastened to a rigid massless bar. The motion of the system in a vertical plane is restrained by two springs of stiffness k and $2k$. For motion in the vertical plane, the system has two degrees of freedom. Let us describe the motion in terms of a translation x and a rotation θ as shown. Assume that the origin of coordinates is at the position of static equilibrium. Since the forces of gravity and the spring forces at the position of static equilibrium just balance each other, we will not have to account for them in writing the equations of motion. For a free vibration of small amplitude, the forces acting on the system are shown in the free-body diagram of Fig. 4-8. As shown, a virtual displacement of the system is represented by a translation δx and a rotation $\delta\theta$. The work done by the inertia and elastic forces is given by

$$\begin{aligned}\delta W &= (-3m\ddot{x} - ml\ddot{\theta} - 3kx - \tfrac{1}{2}kl\theta)\,\delta x \\ &\quad + (-m\ddot{x} - \tfrac{1}{2}ml\ddot{\theta} - \tfrac{1}{2}kx - \tfrac{3}{4}kl\theta)l\,\delta\theta \\ &= 0\end{aligned}$$

Setting the coefficients of δx and $\delta\theta$ to zero, the equations of motion are

$$\sum F_x = -3m\ddot{x} - ml\ddot{\theta} - 3kx - \tfrac{1}{2}kl\theta = 0$$
$$\sum F_\theta = -ml\ddot{x} - \tfrac{1}{2}ml^2\ddot{\theta} - \tfrac{1}{2}klx - \tfrac{3}{4}kl^2\theta = 0$$

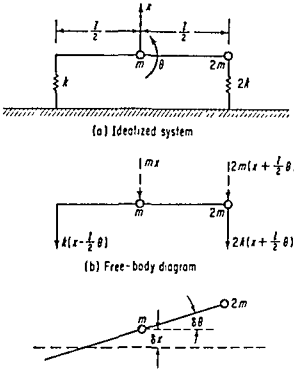


Fig 4-8

Note that the equations are coupled inertially and elastically. The coupling is in both cases symmetric. In matrix form, we can write

$$\begin{Bmatrix} \sum F_x \\ \sum F_\theta \end{Bmatrix} = -\begin{bmatrix} 3m & ml \\ ml & \frac{1}{2}ml^2 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} - \begin{bmatrix} 3k & \frac{1}{2}kl \\ \frac{1}{2}kl & \frac{3}{4}kl^2 \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The generalized inertia forces are given by

$$\begin{Bmatrix} F_x \\ F_\theta \end{Bmatrix}_{in} = -\begin{bmatrix} 3m & ml \\ ml & \frac{1}{2}ml^2 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix}$$

and the generalized elastic forces by

$$\begin{Bmatrix} F_x \\ F_\theta \end{Bmatrix}_{el} = -\begin{bmatrix} 3k & \frac{1}{2}kl \\ \frac{1}{2}kl & \frac{3}{4}kl^2 \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix}$$

The generalized forces associated with x have the dimensions of force while those associated with θ have the dimensions of moment of force

The determinants in the numerators are zero. If the determinant common to both denominators is other than zero, the only possible solution is $A_1 = A_2 = 0$. If the frequency ω is such that this determinant is zero, the amplitudes A are indeterminate and solutions other than the trivial one are possible. Let us set the determinant in the denominator to zero.

$$\begin{vmatrix} 2k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{vmatrix} = 0 \tag{4.28}$$

It is customary to refer to this equation as the characteristic equation of the system. Expansion of the determinant leads to

$$\omega^4 - 3\frac{k}{m}\omega^2 + \frac{k^2}{m^2} = 0 \tag{4.29}$$

Solution of this equation yields the natural frequencies of the system, given by

$$\begin{aligned} \omega_1^2 &= \tfrac{1}{2}(3 - \sqrt{5})\frac{k}{m} \\ \omega_2^2 &= \tfrac{1}{2}(3 + \sqrt{5})\frac{k}{m} \end{aligned} \tag{4.30}$$

There are two natural frequencies, equal in number to the number of degrees of freedom. As a result of adding a mass and a spring to the system of Fig. 2-1, leading to the system of Fig 4-1, we have introduced a natural frequency lower than and one higher than the original natural frequency. This is characteristic of the effect of adding a degree of freedom to a system.

In matrix form, we can write the trial solution as

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \sin \omega t \tag{4.31}$$

Substitution of the trial solution into the equations of motion, Eqs. (4.2), leads to the algebraic equations

$$\begin{bmatrix} 2k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{4.32}$$

The characteristic equation results from setting the determinant of the coefficient matrix to zero.

We are likely to be as interested in the shape of the free vibration as we are in the frequency. Substitution of the first frequency ω_1 into either of Eqs (4.26) yields the amplitude ratio

$$\frac{A_{11}}{A_{21}} = \frac{-1 + \sqrt{5}}{2} \tag{4.33}$$

A second subscript is used to indicate that the amplitudes are associated with

the motion at the first frequency ω_1. Thus one possible solution for the equations of motion is given by the trial solution, Eqs. (4.25), in which the frequency is ω_1 and the amplitudes are related by Eq. (4.33). It is easy to show that the same form of solution results if $\sin \omega t$ is replaced by $\cos \omega t$ in the trial solution. Then we can write the complete solution for a free vibration at the first natural frequency as

$$\begin{aligned} x_1 &= A_{11} \sin \omega_1 t + B_{11} \cos \omega_1 t \\ x_2 &= A_{21} \sin \omega_1 t + B_{21} \cos \omega_1 t \end{aligned} \tag{4.34}$$

in which

$$\frac{A_{11}}{A_{21}} = \frac{B_{11}}{B_{21}} = \frac{-1 + \sqrt{5}}{2} \tag{4.35}$$

The motion described by Eqs. (4.34) and (4.35) represents one of the two principal modes of vibration of the system. It is conventional to refer to the mode of lowest frequency as the fundamental mode.

Substitution of the second natural frequency ω_2 into either of Eqs. (4.26) yields the amplitude ratio

$$\frac{A_{12}}{A_{22}} = \frac{-1 - \sqrt{5}}{2} \tag{4.36}$$

Making use of the arguments which led to Eqs. (4.34) and (4.35), we can write the complete solution for a free vibration at the second natural frequency as

$$\begin{aligned} x_1 &= A_{12} \sin \omega_2 t + B_{12} \cos \omega_2 t \\ x_2 &= A_{22} \sin \omega_2 t + B_{22} \cos \omega_2 t \end{aligned} \tag{4.37}$$

in which

$$\frac{A_{12}}{A_{22}} = \frac{B_{12}}{B_{22}} = \frac{-1 - \sqrt{5}}{2} \tag{4.38}$$

The most general solution for the free vibrations of the system is given by the superposition of the motions in the two principal modes of vibration, given by Eqs. (4.34) and (4.37). We can write

$$\begin{aligned} x_1 &= A_{11} \sin \omega_1 t + B_{11} \cos \omega_1 t + A_{12} \sin \omega_2 t + B_{12} \cos \omega_2 t \\ x_2 &= A_{21} \sin \omega_1 t + B_{21} \cos \omega_1 t + A_{22} \sin \omega_2 t + B_{22} \cos \omega_2 t \end{aligned} \tag{4.39}$$

Considering Eqs. (4.35) and (4.38), we can write the general solution in the alternate form

$$\begin{aligned} x_1 &= \frac{-1 + \sqrt{5}}{2} (C_1 \sin \omega_1 t + C_1' \cos \omega_1 t) \\ &\qquad + \frac{-1 - \sqrt{5}}{2} (C_2 \sin \omega_2 t + C_2' \cos \omega_2 t) \\ x_2 &= C_1 \sin \omega_1 t + C_1' \cos \omega_1 t + C_2 \sin \omega_2 t + C_2' \cos \omega_2 t \end{aligned} \tag{4.40}$$

or, more compactly, in the form

$$\begin{aligned} x_1 &= \phi_{11}q_1 + \phi_{12}q_2 \\ x_2 &= \phi_{21}q_1 + \phi_{22}q_2 \end{aligned} \tag{4.41}$$

in which

$$\begin{aligned} \phi_{11} &= \frac{-1+\sqrt{5}}{2} \qquad & \phi_{12} &= \frac{-1-\sqrt{5}}{2} \\ \phi_{21} &= 1 & \phi_{22} &= 1 \end{aligned} \tag{4.42}$$

and

$$\begin{aligned} q_1 &= C_1 \sin \omega_1 t + C_1' \cos \omega_1 t \\ q_2 &= C_2 \sin \omega_2 t + C_2' \cos \omega_2 t \end{aligned} \tag{4.43}$$

A motion in one of the principal modes having a definite amplitude given by a normalizing condition is referred to as a normal mode of vibration. Note that Eqs. (4.42) represent a normalizing condition as well as a statement of the relative amplitudes given by Eqs. (4.35) and (4.38). As the normalizing condition, the amplitude of the second body has been arbitrarily set equal to unity. The amplitudes of the two bodies in a vibration in the first normal mode are given by $\frac{-1+\sqrt{5}}{2}$ and unity. Similarly, the amplitudes of the two bodies in a vibration in the second normal mode are given by $\frac{-1-\sqrt{5}}{2}$ and unity. The pair of quantities ϕ_{11} and ϕ_{21} characterize the first or fundamental mode shape. Similarly, the pair of quantities ϕ_{12} and ϕ_{22} characterize the second mode shape. The two normal mode shapes are shown in Fig 4-9. The quantities q_1 and q_2 represent the amplitudes of the motion in the two normal modes and are called the normal coordinates In a free vibration, the

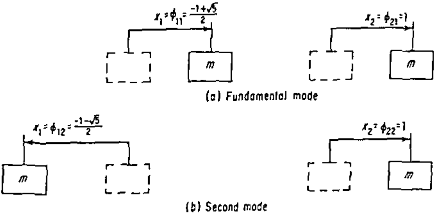


Fig 4-9 *Normal mode shapes*

motion in each of the normal coordinates is a simple harmonic motion at the appropriate natural frequency as indicated by Eqs. (4.43).

We can write the general solution for a free vibration, Eqs. (4.41), in the matrix form

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \tag{4.44}$$

in which, according to Eqs. (4.42), the normal mode shapes are represented by

$$\begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{-1 + \sqrt{5}}{2} & \dfrac{-1 - \sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \tag{4.45}$$

The four constants of integration C_1, C_1', C_2 and C_2' in the solution for the normal coordinates are determined by the nature of the initial disturbance from equilibrium as described by the initial conditions. For illustration, let us assume the initial conditions

$$\begin{aligned} x_1(0) &= \dot{x}_1(0) = \dot{x}_2(0) = 0 \\ x_2(0) &= 1 \end{aligned} \tag{4.46}$$

Substitution of Eqs. (4.46) into Eqs. (4.40) leads to

$$\begin{aligned} C_1 &= C_2 = 0 \\ C_1' &= \frac{1 + \sqrt{5}}{2\sqrt{5}} \\ C_2' &= \frac{-1 + \sqrt{5}}{2\sqrt{5}} \end{aligned} \tag{4.47}$$

Then the free vibrations of the system are described by

$$\begin{aligned} x_1 &= \frac{1}{\sqrt{5}} \cos \omega_1 t - \frac{1}{\sqrt{5}} \cos \omega_2 t \\ x_2 &= \frac{1 + \sqrt{5}}{2\sqrt{5}} \cos \omega_1 t + \frac{-1 + \sqrt{5}}{2\sqrt{5}} \cos \omega_2 t \end{aligned} \tag{4.48}$$

EXAMPLE 4.6

Consider a free vibration of the system of Fig. 4-6. Let us try a solution for the equations of motion having the form

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \sin \omega t$$

Substitution of the trial solution into the equations of motion given in Example 4-3 leads to the algebraic equations

$$\begin{bmatrix} k - I\omega^2 & -k \\ -k & k - I\omega^2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Setting the determinant of the coefficient matrix to zero, we arrive at the characteristic equation

$$\begin{vmatrix} k - I\omega^2 & -k \\ -k & k - I\omega^2 \end{vmatrix} = 0$$

or

$$\omega^4 - 2\frac{k}{I}\omega^2 = 0$$

Solution of the characteristic equation yields the natural frequencies of the system

$$\omega_1 = 0$$
$$\omega_2 = \sqrt{\frac{2k}{I}}$$

Substitution of the natural frequencies into either of the algebraic equations in A_1 and A_2 leads to the amplitude ratios for the two modes, given by

$$\frac{A_{11}}{A_{21}} = 1$$
$$\frac{A_{12}}{A_{22}} = -1$$

The amplitude ratio for the first mode describes a motion with $\theta_1 = \theta_2$. The system moves as a unit without elastic deformation, a motion referred to as a rigid-body mode. We will find that a zero natural frequency is characteristic of a rigid-body mode. The form of the trial solution is not proper if the frequency is zero. Substitution of $\theta_1 = \theta_2$ into the equations of motion results in

$$\ddot{\theta}_1 = \ddot{\theta}_2 = 0$$

having a solution

$$\theta_1 = \theta_2 = C_1 + C_1't$$

where C_1 and C_1' are constants of integration.

For the second mode, the form of the solution remains the same if $\sin \omega t$ is replaced by $\cos \omega t$ in the trial solution. Then the complete solution for a free vibration in the second mode is

$$\theta_1 = -C_2 \sin \omega_2 t - C_2' \cos \omega_2 t$$
$$\theta_2 = C_2 \sin \omega_2 t + C_2' \cos \omega_2 t$$

The general solution for a free vibration of the system results from superposition of the motion in the two modes, leading to

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

where

$$\begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

and

$$\begin{aligned} q_1 &= C_1 + C_1' t \\ q_2 &= C_2 \sin \omega_2 t + C_2' \cos \omega_2 t \end{aligned}$$

As a normalizing condition, we have arbitrarily set the rotation of the second disk equal to unity.

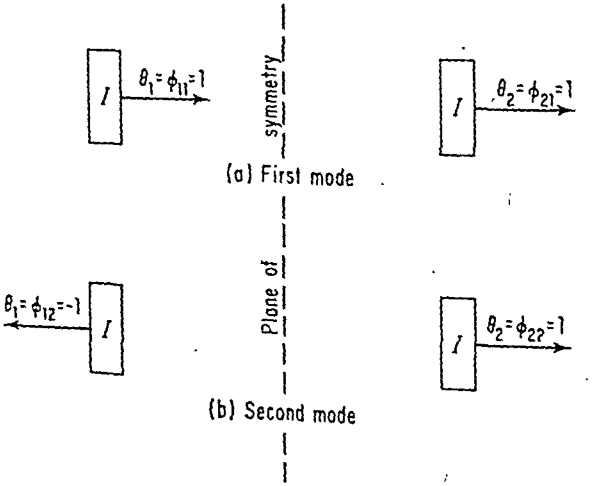


Fig. 4-10 *Normal mode shapes for the system of Fig. 4-6.*

From Fig. 4-10, it is seen that the first and second normal mode shapes are symmetric and antisymmetric with respect to the plane of symmetry of the system. In general, the mode shapes of a system having a plane of symmetry are either symmetric or antisymmetric with respect to the plane of symmetry. Knowing this, we could have written the two mode shapes by inspection. Recognizing that the midpoint of the shaft is stationary in the second mode, we could have obtained the second natural frequency from consideration of the half-system shown in Fig. 4-11. The torsional stiffness of the half shaft is twice that for a full shaft and is thus given by $2k$.

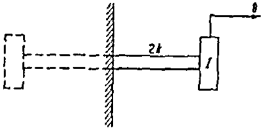


Fig. 4-11 *Model for the second mode of the system of Fig. 4-6.*

EXAMPLE 4.7

Let us determine the nature of a free vibration of the system of Fig 4-8. Substitution of the trial solution

$$\begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} A_x \\ A_\theta \end{Bmatrix} \sin \omega t$$

into the equations of motion given in Example 4.5 leads to the algebraic equations

$$\begin{bmatrix} 3k - 3m\omega^2 & \dfrac{kl}{2} - ml\omega^2 \\ \dfrac{kl}{2} - ml\omega^2 & \dfrac{3kl^2}{4} - \dfrac{ml^2}{2}\omega^2 \end{bmatrix} \begin{Bmatrix} A_x \\ A_\theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The characteristic equation is

$$\begin{vmatrix} 3k - 3m\omega^2 & \dfrac{kl}{2} - ml\omega^2 \\ \dfrac{kl}{2} - ml\omega^2 & \dfrac{3kl^2}{4} - \dfrac{ml^2}{2}\omega^2 \end{vmatrix} = 0$$

leading to

$$\omega^4 - \frac{11}{2}\frac{k}{m}\omega^2 + 4\frac{k^2}{m^2} = 0$$

The two natural frequencies are given by

$$\omega_1^2 = 0.862\frac{k}{m}$$

$$\omega_2^2 = 4.64\frac{k}{m}$$

Substitution of the natural frequencies into either of the equations in A_x and A_θ yields the amplitude ratios for the two modes, given by

$$\frac{A_{x_1}}{A_{\theta_1}} = 0.876l$$

$$\frac{A_{x_2}}{A_{\theta_2}} = -0.379l$$

As a normalizing condition, let us set the rotation θ to unity. Then the normal mode shapes are represented by

$$\begin{bmatrix} \phi_{x_1} & \phi_{x_2} \\ \phi_{\theta_1} & \phi_{\theta_2} \end{bmatrix} = \begin{bmatrix} 0.876l & -0.379l \\ 1 & 1 \end{bmatrix}$$

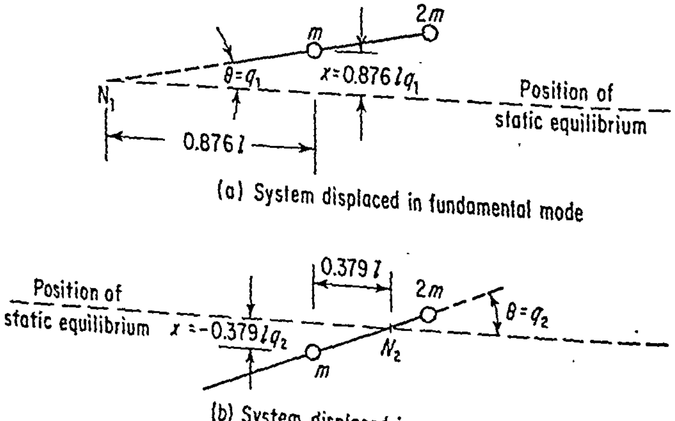
The general solution for a free vibration of the system is given by

$$\begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{bmatrix} 0.876l & -0.379l \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

in which

$$q_1 = C_1 \sin \omega_1 t + C_1' \cos \omega_1 t$$
$$q_2 = C_2 \sin \omega_2 t + C_2' \cos \omega_2 t$$

In Fig. 4-12, the system is shown with small displacements of the normal coordinates q_1 and q_2. For a motion of the fundamental mode, the bar will always be aligned with the point N_1. Similarly, for a motion of the second mode, the point N_2 of the bar will be a stationary point. Points such as N_1 and N_2 are generally referred to as nodal points or nodes.



(a) System displaced in fundamental mode

(b) System displaced in second mode

Fig. 4-12 *Normal mode shapes for the system of Fig. 4-8.*

Suppose the system is set in motion from the position of static equilibrium with the velocity of translation v_0. The initial conditions are given by

$$x(0) = \theta(0) = \dot{\theta}(0) = 0$$
$$\dot{x}(0) = v_0$$

Substitution of the initial conditions into the general solution leads to

$$C_1' = C_2' = 0$$
$$C_1 = 0.796 \frac{v_0}{\omega_1 l}$$
$$C_2 = -0.796 \frac{v_0}{\omega_2 l}$$

For the given initial conditions, the free vibration of the system is

$$x = 0.697 \frac{v_0}{\omega_1} \sin \omega_1 t + 0.302 \frac{v_0}{\omega_2} \sin \omega_2 t$$
$$\theta = 0.796 \frac{v_0}{\omega_1 l} \sin \omega_1 t - 0.796 \frac{v_0}{\omega_2 l} \sin \omega_2 t$$

EXAMPLE 4.8

A uniform rigid bar of mass m and length l is pivoted at one end to a small roller of negligible mass as shown in Fig. 4-13. The roller is free to roll without friction along a horizontal track. If we require the motion of the bar to be planar with the pivot in contact with the track, the system has two degrees of freedom. Let us describe the motion of the bar in terms of the horizontal translation x of the center of mass and the rotation θ of the bar from the vertical.

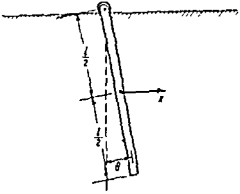


Fig. 4-13

For motion of the system with small amplitude in θ, we can write the kinetic energy as

$$T = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}(\tfrac{1}{12}ml^2)\dot{\theta}^2$$

The generalized inertia forces associated with x and θ are given by

$$F_{x,\mathrm{in}} = -\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = -m\ddot{x}$$
$$F_{\theta,\mathrm{in}} = -\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = -\tfrac{1}{12}ml^2\ddot{\theta}$$

The lack of coupling of the inertia forces results from the choice of the center of mass for the coordinate x.

We can write the potential energy resulting from the gravity force as

$$U = \tfrac{1}{2}mgl(1 - \cos\theta)$$

Then the generalized gravity forces associated with x and θ are

$$F_{x,g} = -\frac{\partial U}{\partial x} = 0$$
$$F_{\theta,g} = -\frac{\partial U}{\partial \theta} = -\tfrac{1}{2}mgl\sin\theta$$

For a motion of small amplitude in θ, we can introduce the approximation $\sin\theta \approx \theta$. According to the principle of dynamic equilibrium

$$\sum F_x = -m\ddot{x} = 0$$
$$\sum F_\theta = -\tfrac{1}{12}ml^2\ddot{\theta} - \tfrac{1}{2}mgl\theta = 0$$

Since the equations of motion are uncoupled, they may be solved easily, resulting in

$$x = C_x + C'_x t$$
$$\theta = C_\theta \sin\omega t + C'_\theta \cos\omega t$$

in which the natural frequency of the vibration in θ is

$$\omega = \sqrt{\frac{6g}{l}}$$

Evidently the first normal mode shape involves a unit horizontal translation of the bar. The second normal mode shape is described as a unit rotation of the bar around the center of mass. The coordinates x and θ are the normal coordinates.

4.6 Transformation to the Normal Coordinates

The motion of the simple system of Fig. 4-1 was described by the translations x_1 and x_2 of the separate masses. Other choices of coordinates are

possible. As an example let us select a coordinate q_1 which represents translation of the two masses as a unit, and a coordinate q_2 which represents translation of the second mass relative to the first. The shapes of the system

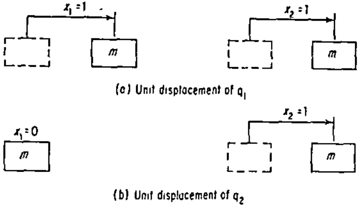


Fig. 4-14 *System shapes corresponding to a particular choice of coordinates q_1, q_2.*

corresponding to displacements of q_1 and q_2 are shown in Fig. 4-14. We can represent the relationship between the pairs of coordinates x_1, x_2 and q_1, q_2 by

$$\begin{aligned} x_1 &= q_1 \\ x_2 &= q_1 + q_2 \end{aligned} \tag{4.49}$$

In matrix form, we can write the equations of transformation as

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \tag{4.50}$$

For a more general choice of coordinates q_1, q_2, we can write the equations of transformation

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \tag{4.51}$$

in which the quantities γ are arbitrary constants. The shapes of the system corresponding to displacements of q_1 and q_2 are shown in Fig. 4-15. A particular choice for the coordinates which is of special importance is that of the normal coordinates. From Eqs. (4.44) and (4.45) we can write the equations of transformation to the normal coordinates q_1 and q_2 in the form

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} \dfrac{-1 + \sqrt{5}}{2} & \dfrac{-1 - \sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \tag{4.52}$$

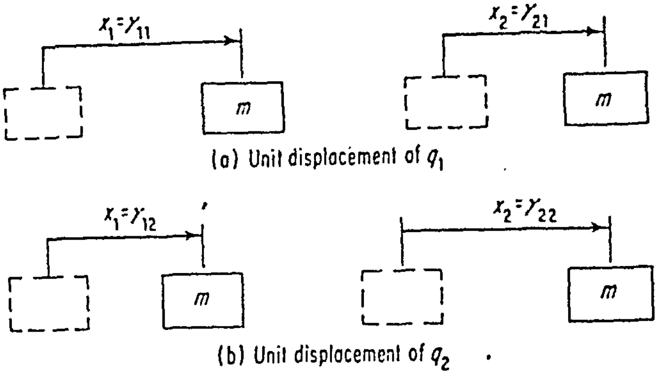


Fig. 4-15 *System shapes corresponding to a more general choice of coordinates* q_1, q_2.

The shapes of the system corresponding to displacements of the normal coordinates are just the normal mode shapes, shown in Fig. 4-9.

Let us write the equations of motion for the system in terms of the normal coordinates. For another choice of coordinates, the procedure would be much the same. We can anticipate that the equations of motion in the normal coordinates can be uncoupled since they represent motion of the normal modes of vibration. The normal modes can exist independently of each other and have no effect on each other. Substitution of the equations of transformation, Eqs. (4.52), into the equations of motion, Eqs. (4.19) and (4.20), leads to

$$\begin{Bmatrix} \sum F_1 \\ \sum F_2 \end{Bmatrix} = -\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \dfrac{-1+\sqrt{5}}{2} & \dfrac{-1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix}$$
$$- \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} \dfrac{-1+\sqrt{5}}{2} & \dfrac{-1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

and to

$$\begin{Bmatrix} \sum F_1 \\ \sum F_2 \end{Bmatrix} = -\begin{bmatrix} \dfrac{-1+\sqrt{5}}{2}m & \dfrac{-1-\sqrt{5}}{2}m \\ m & m \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix}$$
$$- \begin{bmatrix} (-2+\sqrt{5})k & (-2-\sqrt{5})k \\ \dfrac{3-\sqrt{5}}{2}k & \dfrac{3+\sqrt{5}}{2}k \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (4.53)$$

As written, these equations are coupled inertially and elastically and are more complicated than the original equations of motion. Further, the coupling is not symmetric. However, we can find two linear combinations of the equations which will yield uncoupled equations.

As an alternate procedure, we can write the equations of motion using the work and energy approach. From the transpose of Eq. (4.52), the virtual displacements δx_1, δx_2 and δq_1, δq_2 are related by

$$\lfloor \delta x_1 \; \delta x_2 \rfloor = \lfloor \delta q_1 \; \delta q_2 \rfloor \begin{bmatrix} \dfrac{-1+\sqrt{5}}{2} & 1 \\ \dfrac{-1-\sqrt{5}}{2} & 1 \end{bmatrix} \tag{4.54}$$

In Appendix A it is pointed out that the transpose of a matrix product is equal to the product of the transposed matrices taken in inverse order. Substitution of Eqs. (4.53) and (4.54) into Eq. (4.18) leads to

$$\delta W = \lfloor \delta q_1 \; \delta q_2 \rfloor \left\{ -\begin{bmatrix} \dfrac{-1+\sqrt{5}}{2} & 1 \\ \dfrac{-1-\sqrt{5}}{2} & 1 \end{bmatrix} \begin{bmatrix} \dfrac{-1+\sqrt{5}}{2}m & \dfrac{-1-\sqrt{5}}{2}m \\ m & m \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} \right.$$
$$\left. - \begin{bmatrix} \dfrac{-1+\sqrt{5}}{2} & 1 \\ \dfrac{-1-\sqrt{5}}{2} & 1 \end{bmatrix} \begin{bmatrix} (-2+\sqrt{5})k & (-2-\sqrt{5})k \\ \dfrac{3-\sqrt{5}}{2}k & \dfrac{3+\sqrt{5}}{2}k \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \right\} = 0$$

Simplifying, we can write

$$\delta W = \lfloor \delta q_1 \; \delta q_2 \rfloor \begin{Bmatrix} \sum Q_1 \\ \sum Q_2 \end{Bmatrix} = 0$$

in which

$$\begin{Bmatrix} \sum Q_1 \\ \sum Q_2 \end{Bmatrix} = -\begin{bmatrix} \dfrac{5-\sqrt{5}}{2}m & 0 \\ 0 & \dfrac{5+\sqrt{5}}{2}m \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} - \begin{bmatrix} (5-2\sqrt{5})k & 0 \\ 0 & (5+2\sqrt{5})k \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \tag{4.55}$$

Since the virtual displacement is arbitrary, the coefficients of δq_1 and δq_2 must be zero, giving by

$$\begin{Bmatrix} \sum Q_1 \\ \sum Q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{4.56}$$

The equations of motion represented by Eqs. (4.55) and (4.56) are uncoupled and can be solved separately. The generalized inertia and elastic forces associated with q_1 and q_2 are

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}_{in} = -\begin{bmatrix} \dfrac{5-\sqrt{5}}{2}m & 0 \\ 0 & \dfrac{5+\sqrt{5}}{2}m \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix}$$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}_{el} = -\begin{bmatrix} (5-2\sqrt{5})k & 0 \\ 0 & (5+2\sqrt{5})k \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \tag{4.57}$$

These generalized forces could have been obtained from the expressions for the kinetic and potential energies of the system, Eqs. (4.22), if expressed in terms of the coordinates q_1 and q_2.

Evidently the work-and-energy approach represents the most systematic procedure for transforming the equations of motion to the normal coordinates. The uncoupling of the equations of motion will prove to be a significant advantage in dealing with forced vibrations.

EXAMPLE *4.9*

In Example 4.6 the normal mode shapes for the system of Fig. 4-6 were obtained. Making use of the results, we can write the equations of transformation to the normal coordinates q_1 and q_2 as

$$\theta_1 = q_1 - q_2$$
$$\theta_2 = q_1 + q_2$$

The kinetic and potential energies of the system are

$$T = \tfrac{1}{2}I\dot{\theta}_1^2 + \tfrac{1}{2}I\dot{\theta}_2^2$$
$$U = \tfrac{1}{2}k(\theta_2 - \theta_1)^2$$

Substitution of the equations of transformation into the energy expressions leads to

$$T = I\dot{q}_1^2 + I\dot{q}_2^2$$
$$U = 2kq_2^2$$

The generalized inertia and elastic forces associated with q_1 and q_2 are given by

$$Q_{1,in} = -\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = -2I\ddot{q}_1$$
$$Q_{2,in} = -\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = -2I\ddot{q}_2$$

$$Q_{1,el} = -\frac{\partial U}{\partial q_1} = 0$$
$$Q_{2,el} = -\frac{\partial U}{\partial q_2} = -4kq_2$$

Dynamic equilibrium requires that

$$\sum Q_1 = -2I\ddot{q}_1 = 0$$
$$\sum Q_2 = -2I\ddot{q}_2 - 4kq_2 = 0$$

The uncoupled equations of motion can be solved for the natural frequencies, which are

$$\omega_1 = 0$$
$$\omega_2 = \sqrt{\frac{2k}{I}}$$

and the general solution for a free vibration, which is

$$q_1 = C_1 + C_1't$$
$$q_2 = C_2 \sin \omega_2 t + C_2' \cos \omega_2 t$$

These results check with those given in Example 4 6.

EXAMPLE *4.10*

Let us write the equations of motion for the system in Fig 4-8 in terms of the normal coordinates. From Example 4.7 we can write the equations of transformation as

$$x = 0.876lq_1 - 0.379lq_2$$
$$\theta = q_1 + q_2$$

Substitution of the equations of transformation into the virtual work expression given in Example 4 5 results in

$$\begin{aligned} \delta W &= (-4.56ml^2\ddot{q}_1 - 3.93kl^2q_1)\,\delta q_1 \\ &\quad + (-0.173ml^2\ddot{q}_2 - 0.802kl^2q_2)\,\delta q_2 \\ &= 0 \end{aligned}$$

Then the equations of motion for the system can be written as

$$\sum Q_1 = -4.56ml^2\ddot{q}_1 - 3.93kl^2q_1 = 0$$
$$\sum Q_2 = -0.173ml^2\ddot{q}_2 - 0.802kl^2q_2 = 0$$

4 7 Forced Vibrations

Suppose the system of Fig. 4-1 experiences time-variable applied forces $F_1(t)$ and $F_2(t)$ applied as shown in Fig 4-16 In the resulting motion, the forces acting on the bodies are shown on the free-body diagrams of Fig

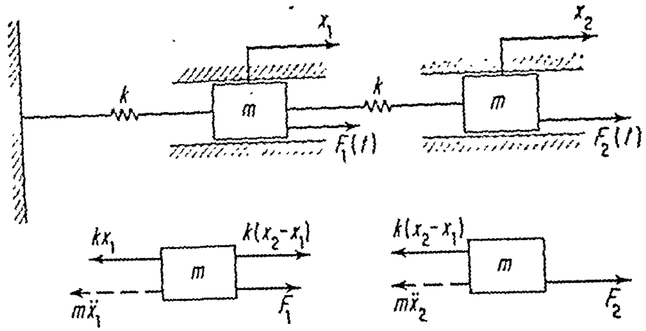


Fig. 4-16 *System with applied force.*

4-16. From the requirements of dynamic equilibrium, we can write the equations of motion as

$$\begin{Bmatrix} \sum F_1 \\ \sum F_2 \end{Bmatrix} = -\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} - \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{4.58}$$

Forced vibration may also result from a motion of the constraints. Imagine that the support has a time-variable motion $x_0(t)$ as shown in Fig. 4-17. Let us measure the coordinates x_1 and x_2 relative to a frame of reference fixed to the support. For the system in motion, the forces acting on the bodies are shown on the free-body diagrams of Fig. 4-17. Then dynamic equilibrium requires that

$$\begin{Bmatrix} \sum F_1 \\ \sum F_2 \end{Bmatrix} = -\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} - \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} - \begin{Bmatrix} m \\ m \end{Bmatrix}\ddot{x}_0 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{4.59}$$

Evidently the forced vibration problems with applied forces or with support motion are identical if $F_1(t) = F_2(t) = -m\ddot{x}_0(t)$.

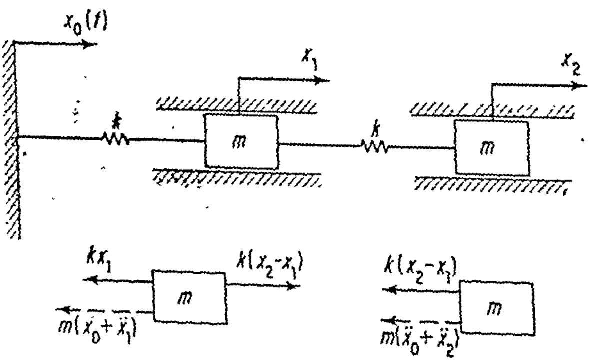


Fig. 4-17 *System with support motion.*

Given the time-history of the external forces or of the support motion, we are faced with the problem of solving a pair of nonhomogeneous ordinary differential equations. Having solved the free vibrations problem, we already have the homogeneous part of the solution. In obtaining the particular solution, we can make use of the superposition principle and the basic solutions for the response to a unit step force, unit impulse, or a harmonic force. Such methods were found to be fruitful in Chap. 3 in dealing with single-degree-of-freedom systems. Generally, however, the basic method for solving the forced vibration problem involves transformation of the equations of motion to new equations in the normal coordinates. The resulting uncoupled equations can be solved separately using the techniques of Chap. 3.

Let us transform the equations of forced vibration, Eqs. (4.58), to a pair of equations in the normal coordinates using the principle of virtual displacements. In Sect. 4.6 we have already transformed the homogeneous portion of the equations of motion, resulting in Eqs. (4.55) and (4.56). The work done by the external forces in a virtual displacement is given by

$$\delta W_{ex} = \lfloor \delta x_1 \; \delta x_2 \rfloor \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \tag{4.60}$$

Substitution of Eq. (4.54) into Eq. (4.60) leads to

$$\delta W_{ex} = \lfloor \delta q_1 \; \delta q_2 \rfloor \begin{bmatrix} \dfrac{-1 + \sqrt{5}}{2} & 1 \\ \dfrac{-1 - \sqrt{5}}{2} & 1 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \tag{4.61}$$

Then the generalized external forces associated with the normal coordinates are

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}_{ex} = \begin{bmatrix} \dfrac{-1 + \sqrt{5}}{2} & 1 \\ \dfrac{-1 - \sqrt{5}}{2} & 1 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \tag{4.62}$$

Inserting this result into Eqs. (4.55) and (4.56), we can write the equations of motion as

$$\begin{Bmatrix} \sum Q_1 \\ \sum Q_2 \end{Bmatrix} = -\begin{bmatrix} \dfrac{5 - \sqrt{5}}{2} m & 0 \\ 0 & \dfrac{5 + \sqrt{5}}{2} m \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix}$$

$$- \begin{bmatrix} (5 - 2\sqrt{5})k & 0 \\ 0 & (5 + 2\sqrt{5})k \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} + \begin{bmatrix} \dfrac{-1 + \sqrt{5}}{2} & 1 \\ \dfrac{-1 - \sqrt{5}}{2} & 1 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{4.63}$$

Given the time history of the external forces F_1 and F_2, this pair of uncoupled equations can be solved using methods outlined in Chap. 3.

EXAMPLE *4.11* The Vibration Absorber

A machine of mass m and mount stiffness k is acted on by a harmonic force $F_0 \sin \omega t$ as shown in Fig. 4-18. The system of mass m' and stiffness k'

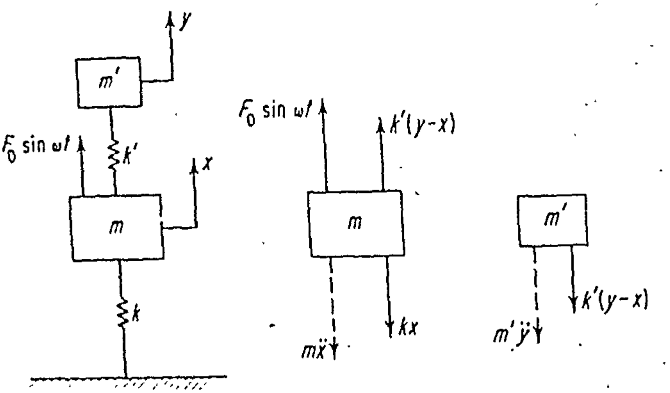


Fig. 4-18 *Harmonic excitation of a simple system with an attached vibration absorber*

mounted to the machine can be tuned as a vibration absorber. From the free-body diagrams we can write the equations of motion for the system as

$$m\ddot{x} + (k + k')x - k'y = F_0 \sin \omega t$$
$$m'\ddot{y} - k'x + k'y = 0$$

The complete solution for these equations is given by the sum of the homogeneous or free vibrations solution and the particular or steady-state solution. Assuming that we are interested in the steady-state solution, we will ignore the free-vibrations part. Since the particular solution is easy to obtain in this case, we will not transform the equations of motion to the normal coordinates. Substitution of the trial solution

$$x = X \sin \omega t$$
$$y = Y \sin \omega t$$

into the equations of motion leads to

$$(k + k' - m\omega^2)X - k'Y = F_0$$
$$-k'X + (k' - m'\omega^2)Y = 0$$

Let us introduce

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\omega_a = \sqrt{\frac{k'}{m'}}$$

$$X_0 = \frac{F_0}{k}$$

in which ω_n and ω_a are the natural frequencies of the machine and the absorber supported separately. X_0 represents the displacement of the machine under the force F_0 applied statically. In terms of these parameters, the equations become

$$\left(1 + \frac{k'}{k} - \frac{\omega^2}{\omega_n^2}\right)X - \frac{k'}{k}Y = X_0$$

$$-X + \left(1 - \frac{\omega^2}{\omega_a^2}\right)Y = 0$$

Examination of the second equation shows the response X of the machine to be zero if the absorber is tuned to $\omega = \omega_a$. Then the first equation yields

$$k'Y = -kX_0 = -F_0$$

Evidently the absorber mass is always out of phase with the applied force and exerts an equal and opposite force on the machine mass

Solving for the response of the machine and absorber, we can write

$$\frac{X}{X_0} = \frac{1 - \dfrac{\omega^2}{\omega_a^2}}{\left(1 + \dfrac{k'}{k} - \dfrac{\omega^2}{\omega_n^2}\right)\left(1 - \dfrac{\omega^2}{\omega_a^2}\right) - \dfrac{k'}{k}}$$

$$\frac{Y}{X_0} = \frac{1}{\left(1 + \dfrac{k'}{k} - \dfrac{\omega^2}{\omega_n^2}\right)\left(1 - \dfrac{\omega^2}{\omega_a^2}\right) - \dfrac{k'}{k}}$$

The response of the machine mass to the applied force is shown in Fig 4-19 as a function of the frequency ratio ω/ω_n for the particular case $\omega_a = \omega_n$ and $m'/m = 0.1$. It follows that $k'/k = 0.1$. Being a two-degree-of-freedom system, there are two resonant frequencies as shown. The resonant frequencies may be obtained by setting the denominator of the response amplitude to zero.

Evidently the vibration absorber will be effective for a constant speed machine having an excitation frequency near the tuned frequency. To soften the effects of the two resonant peaks, it may be desirable to add some damping (Ref. 6) Another example of a vibration absorber is given in Example 5.3.

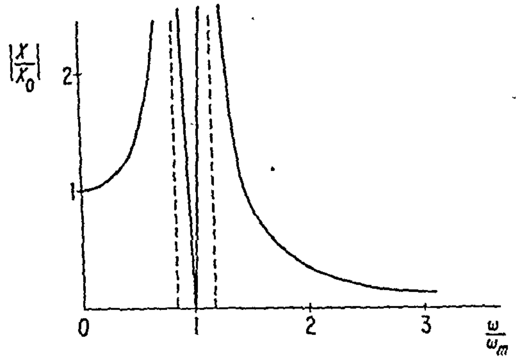


Fig. 4-19 *Response of the system of Fig. 4-18 with $m'/m = k'/k = 0.1$.*

EXAMPLE 4.12

The support of the system of Fig. 4-5 experiences a simple harmonic rotation $\theta_0 \sin \omega t$ as shown in Fig. 4-20. Let us measure the rotations θ_1 and θ_2 relative to the support. Solving the equations of motion for a free vibration given in Example 4.2, the natural frequencies are

$$\omega_1^2 = \tfrac{1}{2}(3 - \sqrt{5})\frac{k}{I}$$
$$\omega_2^2 = \tfrac{1}{2}(3 + \sqrt{5})\frac{k}{I}$$

From the solution for the normal mode shapes, we can write the equations of transformation to the normal coordinates as

$$\theta_1 = \frac{-1 + \sqrt{5}}{2} q_1 + \frac{-1 - \sqrt{5}}{2} q_2$$
$$\theta_2 = q_1 + q_2$$

The kinetic and potential energies of the system are given by

$$T = \tfrac{1}{2}I(\dot{\theta}_1 + \omega\theta_0 \cos \omega t)^2 + \tfrac{1}{2}I(\dot{\theta}_2 + \omega\theta_0 \cos \omega t)^2$$
$$U = \tfrac{1}{2}k\theta_1^2 + \tfrac{1}{2}k(\theta_2 - \theta_1)^2$$

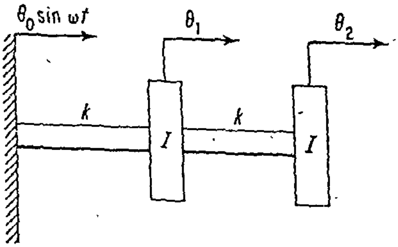


Fig. 4-20

Substitution of the equations of transformation into the energy expressions leads to

$$T = \frac{1}{2}\frac{5-\sqrt{5}}{2} I\dot{q}_1^2 + \frac{1}{2}\frac{5+\sqrt{5}}{2} I\dot{q}_2^2$$
$$+ \tfrac{1}{2}(1+\sqrt{5})I\dot{q}_1\omega\theta_0\cos\omega t + \tfrac{1}{2}(1-\sqrt{5})I\dot{q}_2\omega\theta_0\cos\omega t$$
$$+ I\omega^2\theta_0^2\cos^2\omega t$$
$$U = \tfrac{1}{2}(5-2\sqrt{5})kq_1^2 + \tfrac{1}{2}(5+2\sqrt{5})kq_2^2$$

The generalized inertia and elastic forces associated with q_1 and q_2 are given by

$$Q_{1,in} = -\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = -\frac{5-\sqrt{5}}{2} I\ddot{q}_1 + \frac{1+\sqrt{5}}{2} I\omega^2\theta_0\sin\omega t$$
$$Q_{2,in} = -\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = -\frac{5+\sqrt{5}}{2} I\ddot{q}_2 + \frac{1-\sqrt{5}}{2} I\omega^2\theta_0\sin\omega t$$
$$Q_{1,el} = -\frac{\partial U}{\partial q_1} = -(5-2\sqrt{5})kq_1$$
$$Q_{2,el} = -\frac{\partial U}{\partial q_2} = -(5+2\sqrt{5})kq_2$$

The unknown external torque acting on the shaft at the support will do no work in a virtual displacement of either normal coordinate since the support point represents a nodal point of the normal mode shapes. Thus the associated generalized forces are zero. From the principle of dynamic equilibrium, we can write the equations of motion as

$$\sum Q_1 = -\frac{5-\sqrt{5}}{2} I\ddot{q}_1 - (5-2\sqrt{5})kq_1 + \frac{1+\sqrt{5}}{2} I\omega^2\theta_0\sin\omega t = 0$$
$$\sum Q_2 = -\frac{5+\sqrt{5}}{2} I\ddot{q}_2 - (5+2\sqrt{5})kq_2 + \frac{1-\sqrt{5}}{2} I\omega^2\theta_0\sin\omega t = 0$$

The particular solution for this pair of uncoupled equations represents the steady-state motion of the system. We can write

$$q_1 = \frac{\frac{1+\sqrt{5}}{2} I\omega^2\theta_0}{(5-2\sqrt{5})k - \frac{5-\sqrt{5}}{2} I\omega^2}\sin\omega t$$

$$q_2 = \frac{\frac{1-\sqrt{5}}{2} I\omega^2\theta_0}{(5+2\sqrt{5})k - \frac{5+\sqrt{5}}{2} I\omega^2}\sin\omega t$$

Making use of the expressions for the natural frequencies ω_1 and ω_2, we can simplify the steady-state solution to

$$q_1 = \frac{5}{\ldots} \quad \frac{\omega^2}{\omega_1^2}$$

$$q_2 = \quad \frac{\omega^2}{\omega_2^2}$$

The response of the angl
response of q_1 and q_2 int
been expected, the two re
ω_1 and ω_2.

EXAMPLE *4.13*

A three story building
is represented by a mass

Fig.

floor. Assuming
ing motion, the sh
If the motion is
described by the tra
v hits the third floor an
motion of the building

The forces of impact may be regarded as of short duration. With this assumption, we can consider that the motion of the building is a free vibration resulting from initial conditions which we can determine. If we can assume that the mass m of the airplane is very small with respect to that of the building, we can write the initial conditions as

$$x_1(0) = x_2(0) = \dot{x}_1(0) = 0$$
$$\dot{x}_2(0) = \frac{mv}{M}$$

The momentum of the airplane has been transferred to the third-floor mass.

From the law of motion for a rigid-body translation, we can write the equations of motion for a free vibration as

$$2M\ddot{x}_1 + 3kx_1 - kx_2 = 0$$
$$M\ddot{x}_2 - kx_1 + kx_2 = 0$$

The general solution for the equations of motion is given by

$$x_1 = \tfrac{1}{2}q_1 - q_2$$
$$x_2 = q_1 + q_2$$

in which

$$q_1 = C_1 \sin \omega_1 t + C_1' \cos \omega_1 t$$
$$q_2 = C_2 \sin \omega_2 t + C_2' \cos \omega_2 t$$

The natural frequencies are

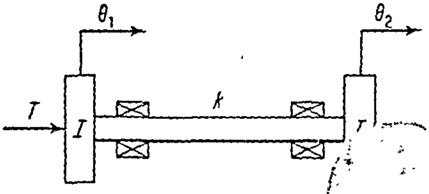
$$\omega_1 = \sqrt{\frac{k}{2M}}$$
$$\omega_2 = \sqrt{\frac{2k}{M}}$$

Substitution of the initial conditions into the general solution leads to

$$C_1 = \frac{2mv}{3M\omega_1}$$
$$C_2 = \frac{mv}{3M\omega_2}$$
$$C_1' = C_2' = 0$$

Then the motion of the building subsequent to the impact is given by

$$x_1 = \frac{mv}{3M\omega_1}(\sin \omega_1 t - \tfrac{1}{2}\sin \omega_2 t)$$
$$x_2 = \frac{2mv}{3M\omega_1}(\sin \omega_1 t + \tfrac{1}{4}\sin \omega_2 t)$$


θ1
θ2
T
I
k
T


T
T0

Dynamic equilibrium requires that

$$\sum Q_1 = -2I\ddot{q}_1 + T = 0$$
$$\sum Q_2 = -2I\ddot{q}_2 - 4kq_2 - T = 0$$

It is most convenient to treat the applied torque as the superposition of two torques with ramp time-histories, the first a positive one applied at $t = 0$ and the second a negative one applied at $t = t_0$. The applied torque can be written as

$$T = \frac{T_0}{t_0} t \qquad \text{for} \quad 0 \le t \le t_0$$
$$\frac{T_0}{t_0} t - \frac{T_0}{t_0}(t - t_0) \qquad \text{for} \quad t_0 \le t$$

If the system is initially at rest, the first equation of motion can easily be integrated, leading to

$$q_1 = \frac{T_0}{12It_0} t^3 \qquad \text{for} \quad 0 \le t \le t_0$$
$$= \frac{T_0}{12It_0}[t^3 - (t - t_0)^3] \qquad \text{for} \quad t_0 \le t$$

The coordinate q_1 represents rigid body rotation of the system.

Let us obtain the response of q_2 using the response to a unit impulse and the principle of superposition. From Eq. (3.50), the response of q_2 to a unit torque impulse is given by

$$q_{2i} = \frac{1}{2I\omega_2} \sin \omega_2 t$$

in which the mass m is replaced by the generalized mass $2I$. Using Eq. (3.57), we can write the response of q_2 to the ramp torque as

$$q_2 = -\int_0^t \frac{T_0}{t_0} t \frac{1}{2I\omega_2} \sin \omega_2(t - t)\, dt$$

From the equation of motion, note that the torque acting is negative. Integration of this expression leads to

$$q_2 = -\frac{T_0}{2I\omega_2^2}\left[\frac{t}{t_0} - \frac{1}{\omega_2 t_0} \sin \omega_2 t\right] \qquad \text{for} \quad 0 \le t \le t_0$$

Adding a ramp torque at $t = t_0$, we can write

$$q_2 = -\frac{T_0}{2I\omega_2^2}\left[\frac{t}{t_0} - \frac{1}{\omega_2 t_0} \sin \omega_2 t\right]$$
$$+ \frac{T_0}{2I\omega_2^2}\left[\frac{t - t_0}{t_0} - \frac{1}{\omega_2 t_0} \sin \omega_2(t - t_0)\right]$$
$$= -\frac{T_0}{2I\omega_2^2}\left[1 - \frac{1}{\omega_2 t_0} \sin \omega_2 t + \frac{1}{\omega_2 t_0} \sin \omega_2(t - t_0)\right] \qquad \text{for} \quad t_0 \le t$$

Since the second natural frequency is given by $\omega_2^2 = 2k/I$, the response of q_2 can be written as

$$q_2 = -\frac{T_0}{4k}\left[\frac{t}{t_0} - \frac{1}{\omega_2 t_0}\sin\omega_2 t\right] \qquad \text{for} \quad 0 \le t \le t_0$$
$$= -\frac{T_0}{4k}\left[1 - \frac{1}{\omega_2 t_0}\sin\omega_2 t + \frac{1}{\omega_2 t_0}\sin\omega_2(t - t_0)\right] \qquad \text{for} \quad t_0 \le t$$

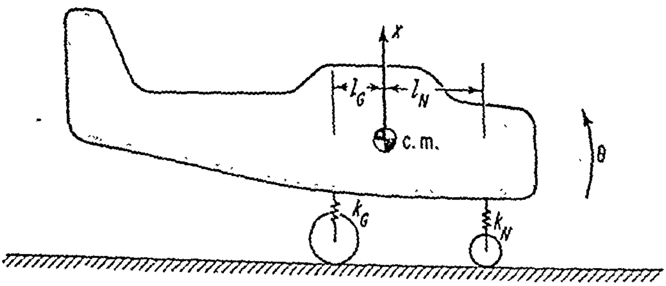
The torque in the shaft is given by

$$k(\theta_2 - \theta_1) = 2kq_2$$
$$= -\frac{T_0}{2}\left[\frac{t}{t_0} - \frac{1}{\omega_2 t_0}\sin\omega_2 t\right] \qquad \text{for } 0 \le t \le t_0$$
$$= -\frac{T_0}{2}\left[1 - \frac{1}{\omega_2 t_0}\sin\omega_2 t + \frac{1}{\omega_2 t_0}\sin\omega_2(t - t_0)\right] \qquad \text{for } t_0 \le t$$

For $\omega_2 t_0 \ll 1$, we expect the response to approach that for a step torque. If the torque builds up gradually, represented by $\omega_2 t_0 \gg 1$, the response approaches that for a statically applied torque.

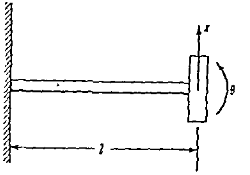
Problems

4-1 The landing gear of an aircraft of mass m is idealized as a pair of springs of total stiffness K_G, representing the main gear, and a spring of stiffness K_N, representing the nose wheel. The main gear, and nose wheel are positioned as shown. The radius of gyration of the aircraft around a lateral axis through the center of mass is r. Write the equations of motion for free vibrations in vertical translation and pitch. What is needed to uncouple the translational and pitching motions?



Prob. 4-1

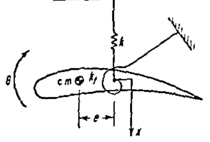
4-2 A disk of mass m and moment of inertia I around a lateral axis through the center of mass is fixed to the end of a uniform cantilever beam as shown. The flexural rigidity of the beam is represented by EI. The mass of the beam is considered to be negligible compared with that of the disk.



Prob. 4-2

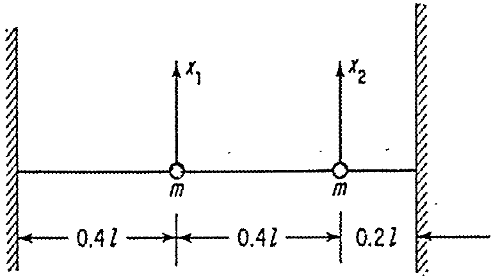
Determine the stiffness and flexibility influence coefficients associated with the coordinates x and θ, which describe translation and rotation of the disk as indicated. Write the equations of motion for a free vibration of the disk in the two forms corresponding to the two sets of influence coefficients.

4-3 An idealized wing is represented by a rigid airfoil section of mass m restrained in translation by a spring of stiffness k and in rotation by a spring of torsional stiffness k_t. The wing section has a radius of gyration r in pitch around the center of mass, which is located a distance e forward of the support point. Write the equations of motion for a free vibration of small amplitude in the coordinates x and θ, using the principle of virtual work. Check the generalized inertia and elastic forces by deriving them from energy expressions.



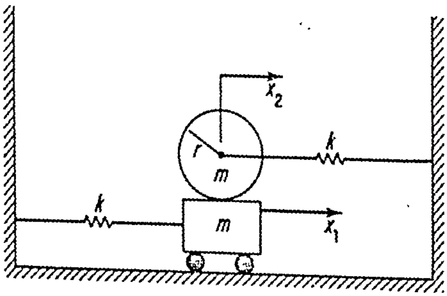
Prob. 4-3

4-4 Two particles of mass m are fastened to an elastic wire which is being stretched with a large tension force T. Determine the stiffness and flexibility influence coefficients associated with the transverse displacements x_1 and x_2. Using the stiffness influence coefficients, write the equations of motion for a free vibration of small amplitude in x_1 and x_2.



Prob. 4-4

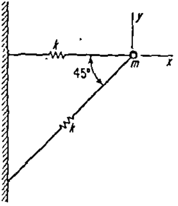
4-5 The homogeneous circular cylinder of mass m rolls without slipping on the slab, also of mass m, which is free to translate horizontally. We can describe the motion in terms of the two coordinates of translation x_1 and x_2. Using the principle of virtual work, write the equations of motion for a free vibration. Check the generalized inertia and elastic forces by deriving them from the energy expressions.



Prob. 4-5

4-6 Determine the natural frequencies and normal mode shapes for the system of Prob. 4-4. A force is applied to the first mass, displacing it by Δ, and then is suddenly released. Find the subsequent motion of the system.

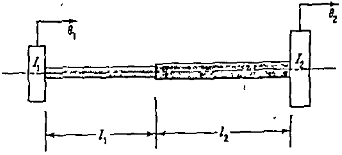
4-7 The particle of mass m is attached to the two springs, each of spring constant k. For a free vibration of small amplitude in the xy plane, determine the natural frequencies and normal mode shapes of the particle. Note that there is a plane of symmetry and that the two natural modes can be described as symmetric and antisymmetric.



Prob. 4-7

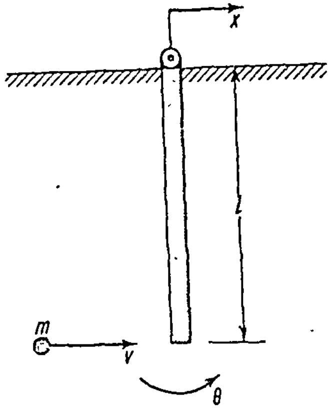
4-8 For a free vibration of small amplitude of the double pendulum of Fig 4-2, determine the natural frequencies and the normal mode shapes. If the point of support is suddenly given a uniform horizontal velocity v, what will be the resulting free vibration?

4-9 Two disks with axial moments of inertia I_1 and I_2 are mou[illegible] shaft as shown. The shaft is made up of two uniform sections, one o[illegible] l_1 and torsional rigidity GJ_1 and the other of length l_2 and torsional [illegible] GJ_2. Determine the natural frequencies and normal [illegible] motion of the system



Prob. 4-9

4-10 A uniform rigid bar of mass m and length l is pivoted to a small roller of negligible mass. The roller is free to roll along the track with negligible friction. The bar experiences a plastic impact at the lower end of a mass m having a velocity v. The mass adheres to the bar. Determine the subsequent free vibration of the system.



Prob. 4-10

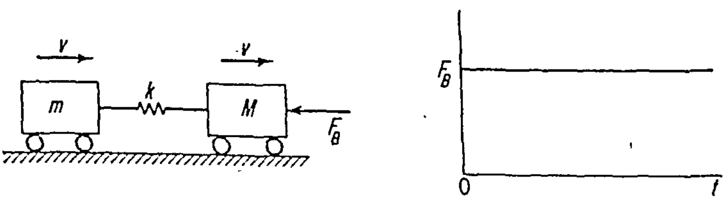
4-11 Determine the natural frequencies for the system of Prob. 4-1. What must be the relationship among l_N, l_G, and r such that the nodal points of the normal modes are at the main gear and at the nose wheel?

4-12 For the double pendulum of Prob. 4-8, transform the equations of motion to an uncoupled pair of equations in the normal coordinates. Use the principle of virtual work.

4-13 Write the equations of motion for a free vibration of the system of Prob. 4-10 in the normal coordinates. Assume that the mass m is fixed to the lower end of the bar. Derive the generalized forces from energy expressions.

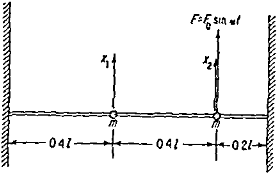
4-14 The aircraft of Probs. 4-1 and 4-11 is taxiing at a velocity v over a taxi strip which has expansion joints with uniform spacing d. For what critical velocities will the disturbances at the expansion joints tend to promote a resonant condition? For simplicity, assume that $r^2 = l_N l_G$.

4-15 A locomotive of mass M is pulling a car of mass m at a velocity v. At an instant, a braking force F_B is applied to the locomotive. Write the equations of motion for the system in the normal coordinates. What is the force experienced by the coupler, represented here by the spring k?



Prob. 4-15

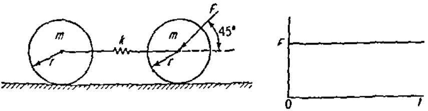
4-16 A harmonic force $F = F_0 \sin \omega t$ acts on the second mass of the system of Probs. 4-4 and 4-6. Write the equations of motion in terms of the normal coordinates. Determine the response of the system.



Prob. 4-16

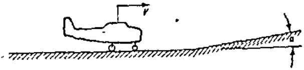
4-17 Show that the results for the response of the system of Example 4.14 reduces to that for a step input if $\omega_2 t_0 \to 0$

4-18 Two homogeneous cylinders of mass m and radius r are connected by a spring of stiffness k as shown. A step force is applied to the axis of one cylinder as indicated Find the peak force in the spring in the resulting motion. The friction force is sufficient to prevent the cylinders from slipping.



Prob. 4-18

4-19 The aircraft of Probs. 4-1 and 4-11 is taxiing at a velocity v when it encounters a ramp which makes a small angle α with the horizontal. Determine the response of the aircraft in vertical translation and pitch. By what



Prob. 4-19

factor is the force in the nosewheel increased compared with that experienced while taxiing? Assume that $r^2 = l_N l_G$.

4-20 For the system of Prob. 4-2 show that the product of the matrix of stiffness influence coefficients $[k]$ and the matrix of flexibility influence coefficients $[C]$ yields the unit matrix.

4-21 Repeat the operations of Prob. 4-12, using matrix notation and algebra.

4-22 Repeat the operations of Prob. 4-16, using matrix notation and algebra.

CHAPTER 5

Free Vibrations of Lumped-Mass Systems with Several Degrees of Freedom

5.1 Introduction

In this chapter we will be involved in a general study of the free vibrations of lumped-mass systems. It will be convenient for us to begin by considering the general form of and the properties of the inertia and elastic restoring forces. The equations of motion for an undamped system are written in a general form. We will examine the nature of the general solution for a free vibration. In particular, we will be concerned with the character of the natural frequencies and the shapes taken by the system in a free vibration. The property of orthogonality of the separate modes of free vibration will be developed and its meaning examined. We will consider the transformation of the equations of motion to the normal coordinates making use of the property of orthogonality. Finally, we will discuss the effect of viscous damping on a free vibration and examine the assumptions that are usually made when the damping forces are small.

5.2 Equations of Motion

(a) *Generalized Elastic Forces*

Let us consider a mechanical system having n degrees of freedom. We can describe the motion of the system with a set of n generalized coordinates $x_1, x_2, \ldots, x_n$. The generalized coordinates are required to be linearly independent and consistent with the constraints placed on the system. The generalized forces associated with the coordinates can be represented by $F_1, F_2, \ldots, F_n$. Let us select for the origin of our coordinates, described by $x_1 = x_2 = \cdots = x_n = 0$, the position of static equilibrium. Then the constant external and elastic forces which may be acting on the system in the equilibrium position can be ignored since they just balance each other. The

elastic forces $F_{1,el}, F_{2,el}, \ldots, F_{n,el}$ corresponding to an arbitrary position of the system represent the change in the elastic forces from those existing at the equilibrium position. Thus $F_{1,el} = F_{2,el} = \cdots = F_{n,el} = 0$ at the position of static equilibrium.

A mechanical system in motion must be in dynamic equilibrium at any instant. If we identify the forces acting on the system as external, inertia, and elastic, we can write

$$\begin{Bmatrix} \sum F_1 \\ \sum F_2 \\ \vdots \\ \sum F_n \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix}_{ex} + \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix}_{in} + \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix}_{el} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \tag{5.1}$$

For convenience, let us refer to the sum of all the forces exclusive of the elastic forces as the applied forces. The applied forces are evidently the negative of the elastic forces. If the applied forces and resulting displacements are related linearly, the force-displacement relationship is given by

$$\begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix}_{ex} + \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix}_{in} = -\begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix}_{el} = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & \cdot \\ \vdots & & & \vdots \\ k_{n1} & \cdot & \cdots & k_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \tag{5.2}$$

The quantities k are just the stiffness influence coefficients introduced in Chap. 4. Recall that the coefficient k_{ij} represents the applied force associated with x_i required to hold the system in a static configuration with unit displacement in x_j and zero displacement in all the other coordinates.

The elastic forces being conservative, the generalized elastic forces can be derived from the elastic potential energy or strain energy. For the forces associated with x_i

$$F_{i,el} = -\frac{\partial U_{el}}{\partial x_i} \tag{5.3}$$

From Eq. (5.2)

$$F_{i,el} = -k_{i1}x_1 - k_{i2}x_2 - \cdots - k_{in}x_n \tag{5.4}$$

Using Eqs. (5.3) and (5.4), we can write

$$\frac{\partial F_{i,el}}{\partial x_j} = -\frac{\partial^2 U_{el}}{\partial x_i\, \partial x_j} = -k_{ij} \tag{5.5}$$

Since the order in which the potential energy is differentiated should make no difference, it is apparent that

$$k_{ij} = k_{ji} = \frac{\partial^2 U_{el}}{\partial x_i\, \partial x_j} \tag{5.6}$$

The symmetry of k is consistent with the reciprocal theorem of Betti.* Further Eq. (5 6) represents a convenient method for deriving the influence coefficients from the strain energy.

An alternate form for the force-displacement relationship of Eq. (5 2) is given by

$$\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & \vdots \\ \vdots & & & \\ C_{n1} & & \cdots & C_{nn} \end{bmatrix} \left\{ \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix}_{ex} + \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix}_{in} \right\} \tag{5.7}$$

in which the quantities C are the flexibility influence coefficients of the system. The coefficient C_{ij} represents the displacement in x_i resulting from application of a unit force associated with x_j. For convenience the applied forces are indicated by

$$\begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix}_{ex} + \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix}_{in} \tag{5.8}$$

Then Eqs. (5 2) and (5.7) can be written in the abbreviated forms

$$\begin{aligned} \{F\} &= [k]\{x\} \\ \{x\} &= [C]\{F\} \end{aligned} \tag{5.9}$$

Substitution of either of Eq (5.9) into the other leads to

$$\begin{aligned} \{F\} &= [k][C]\{F\} \\ \{x\} &= [C][k]\{x\} \end{aligned}$$

Evidently

$$[k][C] = [C][k] = [I] \tag{5.10}$$

where $[I]$ is the unit matrix. The stiffness and flexibility matrices are the inverse of each other, expressed by

$$\begin{aligned} [k] &= [C]^{-1} \\ [C] &= [k]^{-1} \end{aligned} \tag{5.11}$$

Since the stiffness matrix $[k]$ is symmetric, it follows that the flexibility matrix is also symmetric.

* Consult a standard text in mechanics of materials or theory of elasticity Maxwell and Rayleigh are also associated with this theorem

If we express the strain energy in terms of the applied forces $F_1, F_2, \ldots, F_n$, we can derive the displacement x_i from Castigliano's theorem,* written as

$$x_i = \frac{\partial U_{el}}{\partial F_i} \tag{5.12}$$

From Eq. (5.7)

$$x_i = C_{i1}F_1 + C_{i2}F_2 + \cdots + C_{in}F_n \tag{5.13}$$

Using Eqs. (5.12) and (5.13), we can write

$$\frac{\partial x_i}{\partial F_j} = \frac{\partial^2 U_{el}}{\partial F_i\,\partial F_j} = C_{ij} \tag{5.14}$$

Assuming that the order of differentiation of the strain energy makes no difference, we can write

$$C_{ij} = C_{ji} = \frac{\partial^2 U_{el}}{\partial F_i\,\partial F_j} \tag{5.15}$$

Evidently the flexibility matrix is symmetric, as noted earlier.

On some occasions, a mechanical system may be sufficiently unconstrained that motions not involving elastic deformation are possible. For such a rigid-body motion, a displacement $\{x\}$ with at least one nonzero x corresponds to the applied forces $\{F\} = \{0\}$. This situation is not inconsistent with the first of Eqs. (5.9) but it is inconsistent with the second. We conclude that the flexibility matrix $[C]$ cannot be defined for such a system.† For this case, we will find that $[k]$ is singular. Then it is evident from the second of Eqs. (5.11) that $[C]$ cannot be defined.

The elastic potential energy or strain energy stored in a mechanical system is equal to the work done by the applied forces in deforming the system from the position of static equilibrium. We can write

$$\begin{aligned} U_{el} &= \tfrac{1}{2}\lfloor x_1 x_2 \cdots x_n \rfloor \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix} \\ &= \tfrac{1}{2}\lfloor F_1 F_2 \cdots F_n \rfloor \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \end{aligned} \tag{5.16}$$

* Refer to a standard text in mechanics of materials for a development of Castigliano's theorem.

† However, some of the flexibility influence coefficients can be defined for a system with rigid-body motion if the coordinates are chosen properly. For application of this idea to a continuous system, consult Ref. 7.

Substitution of Eqs. (5.2) and (5.7) into the energy expressions leads to two forms for the strain energy, given by

$$
\begin{aligned}
U_{el} &= \tfrac{1}{2}\lfloor x_1 x_2 \cdots x_n \rfloor \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & \cdot \\ \vdots & & & \cdot \\ k_{n1} & & \cdots & k_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \\
&= \tfrac{1}{2}\lfloor F_1 F_2 \cdots F_n \rfloor \begin{bmatrix} C_{11} & C_{12} & \cdot & C_{1n} \\ C_{21} & C_{22} & \cdot & \\ \vdots & & & \vdots \\ C_{n1} & \cdot & \cdot & C_{nn} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix}
\end{aligned}
\tag{5.17}
$$

Expressions having the form of Eqs. (5.17) are referred to as quadratic forms. They are homogeneous since they contain only the second order terms $x_i x_j$ or $F_i F_j$. The strain energy is zero if the system is in the static-equilibrium position. If the equilibrium position is a stable position, the strain energy will be positive for all positions other than the equilibrium position. For this case, the strain energy expressions are referred to as positive definite quadratic forms. *The stiffness and flexibility matrices for this case are referred to as* positive definite.

For systems in which rigid-body motions are possible, the flexibility *matrix [C] cannot be defined and the second of Eqs. (5.17) is inappropriate.* In a rigid-body displacement, a displacement $\{x\}$ with at least one nonzero x is associated with zero strain energy. The system is said to be neutrally stable with respect to the rigid body motions. For this case, the strain energy expression, the first of Eqs. *(5.17), is referred to as a positive semidefinite* quadratic form. The stiffness matrix is referred to as positive semidefinite.

The generalized elastic forces can be derived from the first of the strain energy expressions, Eqs. (5.17). For example, the ith elastic force is given by

$$
\begin{aligned}
F_{i,el} &= -\frac{\partial U_{el}}{\partial x_i} \\
&= -\tfrac{1}{2}\lfloor 0 \cdots \overset{i\text{th term}}{1} \cdots 0 \rfloor \begin{bmatrix} k_{11} & k_{12} & \cdot\ \cdot & k_{1n} \\ k_{21} & k_{22} & \cdot & \cdot \\ \cdot & & & \vdots \\ k_{n1} & \cdot & \cdot\ \cdot & k_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{Bmatrix} \\
&\quad -\tfrac{1}{2}\lfloor x_1 x_2 \cdots x_n \rfloor \begin{bmatrix} k_{11} & k_{12} & \cdot & k_{1n} \\ k_{21} & k_{22} & \cdot & \\ k_{n1} & \cdot & \cdot\ \cdot & k_{nn} \end{bmatrix} \begin{array}{l} \begin{Bmatrix} 0 \\ \cdot \\ 1 \\ \vdots \\ 0 \end{Bmatrix} \end{array} \begin{array}{l} \\ \\ i\text{th term} \\ \\ \\ \end{array}
\end{aligned}
$$

which reduces to

$$F_{1,el} = -\tfrac{1}{2}\lfloor k_{11}k_{12}\cdots k_{1n}\rfloor \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} - \tfrac{1}{2}\lfloor x_1 x_2 \cdots x_n \rfloor \begin{Bmatrix} k_{11} \\ k_{21} \\ \vdots \\ k_{n1} \end{Bmatrix}$$

Let us replace the second term on the right-hand side by its transpose, leading to

$$F_{1,el} = -\tfrac{1}{2}\lfloor k_{11}k_{12}\cdots k_{1n}\rfloor \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} - \tfrac{1}{2}\lfloor k_{11}k_{21}\cdots k_{n1}\rfloor \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$$

Since the matrix $[k]$ is symmetric, the two terms on the right-hand side are identical and we can write

$$F_{1,el} = -\lfloor k_{11}k_{12}\cdots k_{1n}\rfloor \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$$

Then the generalized elastic forces can be summarized by

$$\begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix}_{el} = -\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & \cdot \\ \vdots & & & \vdots \\ k_{n1} & & \cdots & k_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$$

which agrees with Eq. (5.2).

EXAMPLE 5.1

A system is made up of two uniform rigid bars, each of mass m and length l, pinned to each other and supported by springs as shown in Fig. 5-1. Assuming that horizontal motion is not excited, the motion of small amplitude of the system can be described by the translation x and the rotations θ and ϕ. Let us take the origin of coordinates at the static equilibrium position. Then the elastic and inertia forces acting on the system are shown on the free-body diagram. The work done by the elastic forces in a virtual displacement of the system is given by

$$\delta W_{el} = (-6kx + 2kl\theta - 2kl\phi)\,\delta x + (2klx - 2kl^2\theta)\,\delta\theta + (-2klx - 2kl^2\phi)\,\delta\phi$$

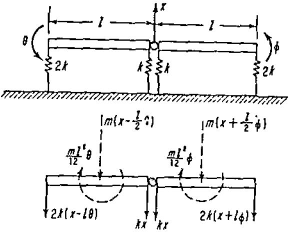


Fig. 5-1

or in matrix form by

$$\delta W_{el} = -\lfloor \delta x \; \delta\theta \; \delta\phi \rfloor \begin{bmatrix} 6k & -2kl & 2kl \\ -2kl & 2kl^2 & 0 \\ 2kl & 0 & 2kl^2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \\ \phi \end{Bmatrix}$$

Since the coefficients of the virtual displacements are by definition the generalized forces, we can write

$$\begin{Bmatrix} F_x \\ F_\theta \\ F_\phi \end{Bmatrix}_{el} = -\begin{bmatrix} 6k & -2kl & 2kl \\ -2kl & 2kl^2 & 0 \\ 2kl & 0 & 2kl^2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \\ \phi \end{Bmatrix}$$

Comparing this result with that of Eq (5.2), the stiffness matrix is given by

$$[k] = \begin{bmatrix} 6k & -2kl & 2kl \\ -2kl & 2kl^2 & 0 \\ 2kl & 0 & 2kl^2 \end{bmatrix}$$

The strain energy of the system is given by

$$\begin{aligned} U_{el} &= \tfrac{1}{2}(2k)(x - l\theta)^2 + \tfrac{1}{2}(2k)x^2 + \tfrac{1}{2}(2k)(x + l\phi)^2 \\ &= \tfrac{1}{2}(6kx^2 + 2kl^2\theta^2 + 2kl^2\phi^2 - 4klx\theta + 4klx\phi) \end{aligned}$$

or in matrix form by

$$U_{el} = \tfrac{1}{2}\lfloor x \; \theta \; \phi \rfloor \begin{bmatrix} 6k & -2kl & 2kl \\ -2kl & 2kl^2 & 0 \\ 2kl & 0 & 2kl^2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \\ \phi \end{Bmatrix}$$

Note that the terms on the diagonal of the square matrix are the coefficients of x^2, θ^2 and ϕ^2. Typical of the off-diagonal terms of the matrix, the sum of the term in the x row and θ column and the term in the θ row and x column is just the coefficient of $x\theta$. By comparison with the first of Eqs. (5.17), it is evident that the square matrix is the stiffness matrix. Instead, the stiffness influence coefficients can be obtained from Eq. (5.6), leading to

$$k_{xx} = \frac{\partial^2 U_{el}}{\partial x^2} = 6k$$

$$k_{x\theta} = k_{\theta x} = \frac{\partial^2 U_{el}}{\partial x\,\partial\theta} = -2kl$$

$$k_{x\phi} = k_{\phi x} = \frac{\partial^2 U_{el}}{\partial x\,\partial\phi} = 2kl$$

$$k_{\theta\theta} = \frac{\partial^2 U_{el}}{\partial\theta^2} = 2kl^2$$

$$k_{\theta\phi} = k_{\phi\theta} = \frac{\partial^2 U_{el}}{\partial\theta\,\partial\phi} = 0$$

$$k_{\phi\phi} = \frac{\partial^2 U_{el}}{\partial\phi^2} = 2kl^2$$

We can construct the stiffness matrix with this result. Having the stiffness matrix, we can write the generalized elastic forces from Eq. (5.2).

As an alternate procedure, we can write the generalized elastic forces using Eq. (5.3) and the energy expression. We can write

$$F_{x,el} = -\frac{\partial U_{el}}{\partial x} = -\tfrac{1}{2}[6k\ \ -2kl\ \ 2kl]\begin{Bmatrix} x \\ \theta \\ \phi \end{Bmatrix} - \tfrac{1}{2}[x\ \theta\ \phi]\begin{Bmatrix} 6k \\ -2kl \\ 2kl \end{Bmatrix}$$

$$F_{\theta,el} = -\frac{\partial U_{el}}{\partial \theta} = -\tfrac{1}{2}[-2kl\ \ 2kl^2\ \ 0]\begin{Bmatrix} x \\ \theta \\ \phi \end{Bmatrix} - \tfrac{1}{2}[x\ \theta\ \phi]\begin{Bmatrix} -2kl \\ 2kl^2 \\ 0 \end{Bmatrix}$$

$$F_{\phi,el} = -\frac{\partial U_{el}}{\partial \phi} = -\tfrac{1}{2}[2kl\ \ 0\ \ 2kl^2]\begin{Bmatrix} x \\ \theta \\ \phi \end{Bmatrix} - \tfrac{1}{2}[x\ \theta\ \phi]\begin{Bmatrix} 2kl \\ 0 \\ 2kl^2 \end{Bmatrix}$$

Replacing each of the second terms on the right-hand side of these equations by its transpose and combining terms, we can summarize the generalized

elastic forces by

$$\begin{Bmatrix} F_x \\ F_\theta \\ \Gamma_\phi \end{Bmatrix}_{el} = -\begin{bmatrix} 6k & -2kl & 2kl \\ -2kl & 2kl^2 & 0 \\ 2kl & 0 & 2kl^2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \\ \phi \end{Bmatrix}$$

In writing the matrix form for the strain energy, the square matrix need not be a symmetric matrix. For example, the potential energy is expressed equally well by

$$U_{el} = \tfrac{1}{2}[x\ \theta\ \phi] \begin{bmatrix} 6k & 0 & 4kl \\ -4kl & 2kl^2 & 0 \\ 0 & 0 & 2kl^2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \\ \phi \end{Bmatrix}$$

as can be verified by performing the multiplication. However, it is a convenience to require the square matrix to be symmetric and this will be our practice.

(b) *Generalized Inertia Forces*

As a result of its definition, the kinetic energy of a mechanical system is a positive definite quadratic form in the generalized velocities $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n$. We can write

$$T = \tfrac{1}{2}[\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n] \begin{bmatrix} m_{11} & m_{12} & \cdot & m_{1n} \\ m_{21} & m_{22} & \cdot & \cdot \\ m_{n1} & \cdot & & m_{nn} \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_n \end{Bmatrix} \tag{5.18}$$

The inertia coefficients m are in general functions of the displacements x, expressed by

$$m_{ij} = m_{ij}(x_1, x_2, \ldots, x_n) \tag{5.19}$$

For small amplitudes of the motion, it is usual to approximate the coefficients by their value at the origin, the static equilibrium position, as indicated by

$$m_{ij} = m_{ij}(0, 0, \ldots, 0) \tag{5.20}$$

A system in which the dependence of the inertia coefficients on the displacements must be considered is examined in Example 5.3. The inertia matrix $[m]$ need not be written as a symmetric matrix. For convenience, however, we will always require it to be symmetric, expressed by $m_{ij} = m_{ji}$.

The generalized inertia forces can be derived from the kinetic energy expression. For the ith inertia force

$$F_{i,in} = -\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) + \frac{\partial T}{\partial x_i} \tag{5.21}$$

If we can regard the inertia coefficients as constants, as in Eq. (5.20), the second term on the right-hand side of Eq. (5.21) will be zero. Then, making use of the assumed symmetry of the inertia matrix, we can write

$$\frac{\partial T}{\partial \dot{x}_i} = \lfloor m_{i1} m_{i2} \cdots m_{in} \rfloor \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{Bmatrix}$$

The ith generalized inertia force is given by

$$F_{i,in} = -\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) = -\lfloor m_{i1} m_{i2} \cdots m_{in} \rfloor \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \vdots \\ \ddot{x}_n \end{Bmatrix} \tag{5.22}$$

We can summarize the inertia forces acting on the system by

$$\begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix}_{in} = -\begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & \vdots \\ \vdots & & & \vdots \\ m_{n1} & \cdot & \cdots & m_{nn} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \vdots \\ \ddot{x}_n \end{Bmatrix} \tag{5.23}$$

From Eqs. (5.2) and (5.23), we can write the equations of motion, Eq. (5.1), as

$$\begin{Bmatrix} \sum F_1 \\ \sum F_2 \\ \vdots \\ \sum F_n \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix}_{ex} - \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & \cdot \\ \vdots & & & \vdots \\ m_{n1} & \cdot & \cdots & m_{nn} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \vdots \\ \ddot{x}_n \end{Bmatrix}$$

$$- \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & \cdot \\ \vdots & & & \vdots \\ k_{n1} & \cdot & \cdots & k_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \tag{5.24}$$

Generally these equations are coupled inertially and elastically. However, in most applications the equations are not inertially coupled and the inertia matrix is diagonal.

EXAMPLE 5.2

Let us consider again the system of Fig. 5-1. Referring to the free-body diagram, the work done by the inertia forces and moments in a virtual displacement of the system is given by

$$\delta W_{in} = (-2m\ddot{x} + \tfrac{1}{2}ml\ddot{\theta} - \tfrac{1}{2}ml\ddot{\phi})\,\delta x + (\tfrac{1}{2}ml\ddot{x} - \tfrac{1}{3}ml^2\ddot{\theta})\,\delta\theta + (-\tfrac{1}{2}ml\ddot{x} - \tfrac{1}{3}ml^2\ddot{\phi})\,\delta\phi$$

or in matrix form by

$$\delta W_{in} = -\lfloor \delta x\ \delta\theta\ \delta\phi \rfloor \begin{bmatrix} 2m & -\frac{1}{2}ml & \frac{1}{2}ml \\ -\frac{1}{2}ml & \frac{1}{3}ml^2 & 0 \\ \frac{1}{2}ml & 0 & \frac{1}{3}ml^2 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \\ \ddot{\phi} \end{Bmatrix}$$

By definition, the coefficients of the virtual displacements are the generalized forces. Then the generalized inertia forces are

$$\begin{Bmatrix} F_x \\ F_\theta \\ F_\phi \end{Bmatrix}_{in} = -\begin{bmatrix} 2m & -\frac{1}{2}ml & \frac{1}{2}ml \\ -\frac{1}{2}ml & \frac{1}{3}ml^2 & 0 \\ \frac{1}{2}ml & 0 & \frac{1}{3}ml^2 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \\ \ddot{\phi} \end{Bmatrix}$$

From Eq. (5.23), it is evident that the inertia matrix is just

$$[m] = \begin{bmatrix} 2m & -\frac{1}{2}ml & \frac{1}{2}ml \\ -\frac{1}{2}ml & \frac{1}{3}ml^2 & 0 \\ \frac{1}{2}ml & 0 & \frac{1}{3}ml^2 \end{bmatrix}$$

The kinetic energy for the system is given by

$$\begin{aligned} T &= \tfrac{1}{2}m(\dot{x} - \tfrac{1}{2}l\dot{\theta})^2 + \frac{1}{2}\left(\frac{m}{12}\right)l^2\dot{\theta}^2 + \tfrac{1}{2}m(\dot{x} + \tfrac{1}{2}l\dot{\phi})^2 + \frac{1}{2}\left(\frac{m}{12}\right)l^2\dot{\phi}^2 \\ &= \tfrac{1}{2}(2m\dot{x}^2 + \tfrac{1}{3}ml^2\dot{\theta}^2 + \tfrac{1}{3}ml^2\dot{\phi}^2 - ml\dot{x}\dot{\theta} + ml\dot{x}\dot{\phi}) \end{aligned}$$

or in matrix form by

$$T = \tfrac{1}{2}\lfloor \dot{x}\ \dot{\theta}\ \dot{\phi} \rfloor \begin{bmatrix} 2m & -\frac{1}{2}ml & \frac{1}{2}ml \\ -\frac{1}{2}ml & \frac{1}{3}ml^2 & 0 \\ \frac{1}{2}ml & 0 & \frac{1}{3}ml^2 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{\theta} \\ \dot{\phi} \end{Bmatrix}$$

By comparison with Eq. (5.18), the square matrix is seen to be the inertia matrix $[m]$ given above. Given the inertia matrix, we can write the generalized inertia forces using Eq. (5.23).

The derivatives of the kinetic energy with respect to the generalized velocities are

$$\frac{\partial T}{\partial \dot{x}} = \left[2m \quad -\tfrac{1}{2}ml \quad \tfrac{1}{2}ml\right]\begin{Bmatrix}\dot{x}\\ \dot{\theta}\\ \dot{\phi}\end{Bmatrix}$$

$$\frac{\partial T}{\partial \dot{\theta}} = \left[-\tfrac{1}{2}ml \quad \tfrac{1}{3}ml^2 \quad 0\right]\begin{Bmatrix}\dot{x}\\ \dot{\theta}\\ \dot{\phi}\end{Bmatrix}$$

$$\frac{\partial T}{\partial \dot{\phi}} = \left[\tfrac{1}{2}ml \quad 0 \quad \tfrac{1}{3}ml^2\right]\begin{Bmatrix}\dot{x}\\ \dot{\theta}\\ \dot{\phi}\end{Bmatrix}$$

Then the generalized inertia forces are given by

$$\begin{Bmatrix}F_x\\ F_\theta\\ F_\phi\end{Bmatrix}_{in} = -\frac{d}{dt}\begin{Bmatrix}\dfrac{\partial T}{\partial \dot{x}_1}\\ \dfrac{\partial T}{\partial \dot{\theta}_2}\\ \dfrac{\partial T}{\partial \dot{\phi}_3}\end{Bmatrix} = -\begin{bmatrix}2m & -\tfrac{1}{2}ml & \tfrac{1}{2}ml\\ -\tfrac{1}{2}ml & \tfrac{1}{3}ml^2 & 0\\ \tfrac{1}{2}ml & 0 & \tfrac{1}{3}ml^2\end{bmatrix}\begin{Bmatrix}\ddot{x}\\ \ddot{\theta}\\ \ddot{\phi}\end{Bmatrix}$$

which is in agreement with the previous results.

EXAMPLE 5.3

A pendulum vibration absorber of mass m is attached to a rotating disk as shown in Fig. 5-2. The axial moment of inertia of the disk is represented by I. We can describe the motion of the system in terms of the angles θ_1

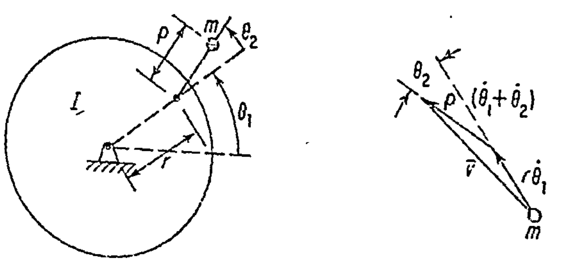


Fig. 5-2 *Pendulum vibration absorber.*

and θ_2 as shown. The angular velocity $\dot{\theta}_1$ of the disk is not to be considered small. Oscillation of the pendulum in θ_2 will evidently have an effect on the angular velocity $\dot{\theta}_1$. The kinetic energy of the system is

$$T = \tfrac{1}{2}I\dot{\theta}_1^2 + \tfrac{1}{2}m\bar{v}\cdot\bar{v}$$

Making use of the vector diagram of the velocity $\bar{v}$ shown in Fig. 5-2, we can write

$$T = \tfrac{1}{2}I\dot{\theta}_1^2 + \tfrac{1}{2}m[r^2\dot{\theta}_1^2 + \rho^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2r\rho\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)\cos\theta_2]$$

In matrix form

$$T = \tfrac{1}{2}\lfloor\dot{\theta}_1\ \dot{\theta}_2\rfloor\begin{bmatrix} I + m(r^2 + \rho^2 + 2r\rho\cos\theta_2) & m\rho(\rho + r\cos\theta_2) \\ m\rho(\rho + r\cos\theta_2) & m\rho^2 \end{bmatrix}\begin{Bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{Bmatrix}$$

Note that the inertia coefficients have the form of Eq. (5.19) in that three of them are functions of θ_2.

Let us obtain the generalized inertia forces from the kinetic energy expression. We can write

$$\frac{\partial T}{\partial\dot{\theta}_1} = \lfloor I + m(r^2 + \rho^2 + 2r\rho\cos\theta_2)\ m\rho(\rho + r\cos\theta_2)\rfloor\begin{Bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{Bmatrix}$$

$$\frac{\partial T}{\partial\theta_1} = 0$$

$$\frac{\partial T}{\partial\dot{\theta}_2} = \lfloor m\rho(\rho + r\cos\theta_2)\ m\rho^2\rfloor\begin{Bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{Bmatrix}$$

$$\frac{\partial T}{\partial\theta_2} = \tfrac{1}{2}\lfloor\dot{\theta}_1\ \dot{\theta}_2\rfloor\begin{bmatrix} -2mr\rho\sin\theta_2 & -mr\rho\sin\theta_2 \\ -mr\rho\sin\theta_2 & 0 \end{bmatrix}\begin{Bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{Bmatrix}$$

Further

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}_1}\right) &= \lfloor -2mr\rho\dot{\theta}_2\sin\theta_2,\ -mr\rho\dot{\theta}_2\sin\theta_2\rfloor\begin{Bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{Bmatrix} \\ &\quad + \lfloor I + m(r^2 + \rho^2 + 2r\rho\cos\theta_2),\ m\rho(\rho + r\cos\theta_2)\rfloor\begin{Bmatrix}\ddot{\theta}_1 \\ \ddot{\theta}_2\end{Bmatrix}\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}_2}\right) &= \lfloor -mr\rho\dot{\theta}_2\sin\theta_2,\ 0\rfloor\begin{Bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{Bmatrix} \\ &\quad + \lfloor m\rho(\rho + r\cos\theta_2),\ m\rho^2\rfloor\begin{Bmatrix}\ddot{\theta}_1 \\ \ddot{\theta}_2\end{Bmatrix}\end{aligned}$$

Let us represent the angular velocity of the disk by

$$\dot{\theta}_1 = \Omega + \dot{\phi}_1$$

in which Ω is the average angular velocity and $\dot{\phi}_1$ is the variation resulting from the motion of the pendulum. Assuming the amplitude θ_2 of the motion of the pendulum to be small, we can write

$$\sin\theta_2 \approx \theta_2$$
$$\cos\theta_2 \approx 1$$

Further we can expect the quantities $\frac{\dot{\phi}_1}{\Omega}, \frac{\dot{\theta}_2}{\Omega}, \frac{\ddot{\phi}_1}{\Omega^2}$, and $\frac{\ddot{\theta}_2}{\Omega^2}$ to be small. Ignoring products of small quantities, we can rewrite certain of the expressions above as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = \lfloor I + m(r+\rho)^2, m\rho(r+\rho)\rfloor \begin{Bmatrix} \ddot{\phi}_1 \\ \ddot{\theta}_2 \end{Bmatrix}$$

$$\frac{\partial T}{\partial \theta_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) = \lfloor m\rho(r+\rho), m\rho^2\rfloor \begin{Bmatrix} \ddot{\phi}_1 \\ \ddot{\theta}_2 \end{Bmatrix}$$

$$\frac{\partial T}{\partial \theta_2} = -mr\rho\Omega^2\theta_2$$

Then the generalized inertia forces are given by

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}_{\text{in}} = -\begin{bmatrix} I + m(r+\rho)^2 & m\rho(r+\rho) \\ m\rho(r+\rho) & m\rho^2 \end{bmatrix} \begin{Bmatrix} \ddot{\phi}_1 \\ \ddot{\theta}_2 \end{Bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & mr\rho\Omega^2 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \theta_2 \end{Bmatrix}$$

Since no other forces are acting on the system, the equations of motion for the system are given by

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}_{\text{in}} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Note that the natural frequency of the pendulum will be proportional to the angular velocity Ω. In machinery, the disturbing torques are also generally proportional to the angular velocity Ω. Thus this type of vibration absorber can be tuned conveniently to absorb the effect of the disturbing torques.

(c) *Other Motion-Dependent Forces*

In addition to the elastic and inertia forces considered, there may be other forces acting which depend on the motion. As an example, all real systems

experience damping forces. We will discuss the effect of viscous damping in Sec. 5.5. Although viscous damping is the simplest form of damping to consider, we will find that it makes the analysis much more difficult.

A damped system is of course nonconservative. Velocity-dependent forces can also appear in the equations of motion for a conservative system. If we consider the vibrations of small amplitude of a system containing a high speed rotor, we will find forces involving the velocities. These gyroscopic forces are actually inertia forces, represented by the second term on the right-hand side of Eq. (5.21). The presence of gyroscopic terms in the equations of motion complicates the analysis in much the same way as does the presence of viscous damping terms.

Sometimes a system experiences forces at the boundaries which depend on the motion of the boundary. If there is an interchange of energy with the surroundings, the system will in general be nonconservative. As an example, consider the oscillation of an airfoil in a fluid flow. It is not convenient to include the fluid as part of the system. However, the fluid will exert forces on the airfoil which depend on the displacements, velocities and accelerations of the airfoil This problem is important to the study of the stability and control of aircraft or of aeroelastic behavior.

For further discussion of nonconservative systems, the reader is referred to Ref. 8.

5.3 Solution of the Equations of Motion

(a) The General Solution for Free Vibrations

From Eq. (5.24), the equations of motion for a free vibration of the n-degree-of-freedom system are given by

$$[m]\{\ddot{x}\} + [k]\{x\} = \{0\} \tag{5.25}$$

Let us seek a solution for the equations of motion having the form

$$\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{Bmatrix} \sin \omega t \tag{5.26}$$

in which $\{A\}$ represents the shape and amplitude of vibration and ω represents the frequency. Substitution of the trial solution into the equations of motion leads to

$$[[k] - \omega^2[m]]\{A\} = \{0\} \tag{5.27}$$

For a nontrivial solution for the amplitudes $\{A\}$, the determinant of the coefficients must be zero, a condition which results in the characteristic equation

$$|[k] - \omega^2[m]| = 0 \tag{5.28}$$

This is an algebraic equation of the nth order in the quantities ω^2, referred to as the characteristic values or eigenvalues of the system.

Since the inertia and stiffness matrices $[m]$ and $[k]$ are real and symmetric, it can be shown that the eigenvalues ω^2 are real. The inertia matrix is always positive definite. If the stiffness matrix is also positive definite, the eigenvalues will be positive and nonzero. Recall that the stiffness matrix is positive definite if the position of static equilibrium is stable. For this case, the natural frequencies ω will be real and nonzero. If the stiffness matrix is positive semidefinite, the eigenvalues will be positive or zero. The stiffness matrix is positive semidefinite if rigid-body motions are possible. For this case, the natural frequencies will be real but may be zero. From experience in working problems, we should recognize that the natural frequency associated with each of the rigid-body degrees of freedom will be zero. It is customary to arrange the natural frequencies $\omega_1, \omega_2, \ldots, \omega_n$ in the order of increasing magnitude.

Having obtained any one of the eigenvalues, we can determine from Eq. (5.27) the shape of the corresponding free vibration represented by the amplitudes $\{A\}$. Let us consider the case in which the eigenvalues are all distinct and nonzero. We will have determined the shape of the motion if we solve for $n - 1$ of the amplitudes in terms of the remaining amplitude. To accomplish this, we will need to solve $n - 1$ of the n equations given by Eq. (5.27). For a distinct eigenvalue, the coefficient matrix of Eq. (5.27) can be shown to be of rank $n - 1$. The rank of a matrix represents the largest nonzero determinant which can be formed from any of the rows and columns. Suppose the first $n - 1$ rows and columns forms a nonzero determinant. Partitioning the matrices of Eq. (5.27), we can write for the kth eigenvalue ω_k^2 that

$$\left[\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1,n-1} \\ k_{21} & k_{22} & \cdots & \cdot \\ \vdots & & & \vdots \\ k_{n-1,1} & \cdot & \cdots & k_{n-1,n-1} \end{bmatrix} - \omega_k^2 \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1,n-1} \\ m_{21} & m_{22} & \cdots & \cdot \\ \vdots & & \cdot & \vdots \\ m_{n-1,1} & \cdot & \cdots & m_{n-1,n-1} \end{bmatrix}\right]$$

$$\times \begin{Bmatrix} A_{1k} \\ A_{2k} \\ \vdots \\ A_{n-1,k} \end{Bmatrix} + \begin{Bmatrix} k_{1n} - m_{1n}\omega_k^2 \\ k_{2n} - m_{2n}\omega_k^2 \\ \vdots \\ k_{n-1,n} - m_{n-1,n}\omega_k^2 \end{Bmatrix} A_{nk} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}$$

The nth row of the coefficient matrix has been dropped. Since we have assumed the coefficient matrix to be nonsingular, we can solve for the first $n - 1$ amplitudes $A_{1k}, A_{2k}, \ldots, A_{n-1,k}$ in terms of the nth amplitude A_{nk}, given by

$$\begin{Bmatrix} A_{1k} \\ A_{2k} \\ \vdots \\ A_{n-1,k} \end{Bmatrix} = -\left[\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1,n-1} \\ k_{21} & k_{22} & \cdots & \cdot \\ \cdot & & & \\ k_{n-1,1} & & \cdots & k_{n-1,n-1} \end{bmatrix} - \omega_k^2 \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1,n-1} \\ m_{21} & m_{22} & \cdots & \\ \vdots & & & \cdot \\ m_{n-1,1} & \cdot & \cdots & m_{n-1,n-1} \end{bmatrix}\right]^{-1} \times \begin{Bmatrix} k_{1n} - m_{1n}\omega_k^2 \\ k_{2n} - m_{2n}\omega_k^2 \\ \vdots \\ k_{n-1,n} - m_{n-1,n}\omega_k^2 \end{Bmatrix} A_{nk} \tag{5.29}$$

If the first $n - 1$ rows and columns of the coefficient matrix do not form a nonsingular matrix, we can always form one from another set of $n - 1$ rows and columns. The column matrix resulting from the matrix product on the right-hand side of Eq (5.29) represents the ratios of the first $n - 1$ amplitudes with the nth amplitude. We can rewrite Eq. (5.29) in the form

$$\begin{Bmatrix} A_{1k} \\ \cdot \\ \dfrac{A_{n-1,k}}{A_{nk}} \end{Bmatrix} = \begin{Bmatrix} \dfrac{A_{1k}}{A_{nk}} \\ \vdots \\ \dfrac{A_{n-1,k}}{A_{nk}} \\ 1 \end{Bmatrix} A_{nk} \tag{5.30}$$

or in the form

$$\begin{Bmatrix} A_{1k} \\ A_{2k} \\ \vdots \\ A_{nk} \end{Bmatrix} = \begin{Bmatrix} \phi_{1k} \\ \phi_{2k} \\ \vdots \\ \phi_{nk} \end{Bmatrix} C_k \tag{5.31}$$

in which the matrix $\{\phi\}_k$ represents the column matrix on the right-hand side of Eq. (5.30) multiplied by an arbitrary constant. The matrix $\{\phi\}_k$ represents the shape of the kth normal mode of vibration. As a normalizing condition, the magnitude of $\{\phi\}_k$ is often chosen such that the largest element is unity. Of course any multiple of $\{\phi\}_k$ will serve to describe the kth normal mode shape. The constant C_k represents the undetermined amplitude in the

harmonic motion $\sin \omega_k t$. The n numbers in $\{\phi\}_k$ may be thought of as the components of a vector in an n-dimensional space. Thus the matrix $\{\phi\}_k$ is often referred to as the kth characteristic vector or eigenvector of the system.

A solution having the same form as that given above results if $\sin \omega t$ is replaced by $\cos \omega t$. Thus we can write the solution for a free vibration in the kth normal mode as

$$\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} \phi_{1k} \\ \phi_{2k} \\ \vdots \\ \phi_{nk} \end{Bmatrix} (C_k \sin \omega_k t + C_k' \cos \omega_k t) \tag{5.32}$$

The general solution for a free vibration of the system results from superposition of the motions in all of the normal modes, leading to

$$\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \cdot \\ \vdots & & & \vdots \\ \phi_{n1} & \cdot & \cdots & \phi_{nn} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{Bmatrix} \tag{5.33}$$

in which

$$\begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{Bmatrix} = \begin{Bmatrix} C_1 \sin \omega_1 t + C_1' \cos \omega_1 t \\ C_2 \sin \omega_2 t + C_2' \cos \omega_2 t \\ \vdots \\ C_n \sin \omega_n t + C_n' \cos \omega_n t \end{Bmatrix} \tag{5.34}$$

The constants in the solution are determined by the initial conditions imposed on the system.

EXAMPLE 5.4

Let us determine the normal modes of free vibration of the system of Fig. 5-1. Making use of the generalized elastic and inertia forces derived in Examples 5.1 and 5.2, we can write the equations of motion as

$$\begin{bmatrix} 2m & -\frac{1}{2}ml & \frac{1}{2}ml \\ -\frac{1}{2}ml & \frac{1}{3}ml^2 & 0 \\ \frac{1}{2}ml & 0 & \frac{1}{3}ml^2 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} 6k & -2kl & 2kl \\ -2kl & 2kl^2 & 0 \\ 2kl & 0 & 2kl^2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \\ \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Substitution of the trial solution

$$\begin{Bmatrix} x \\ \theta \\ \phi \end{Bmatrix} = \begin{Bmatrix} A_x \\ A_\theta \\ A_\phi \end{Bmatrix} \sin \omega t$$

into the equations of motion leads to the algebraic equations

$$\left[\begin{bmatrix} 6k & -2kl & 2kl \\ -2kl & 2kl^2 & 0 \\ 2kl & 0 & 2kl^2 \end{bmatrix} - \omega^2 \begin{bmatrix} 2m & -\frac{1}{2}ml & \frac{1}{2}ml \\ -\frac{1}{2}ml & \frac{1}{3}ml^2 & 0 \\ \frac{1}{2}ml & 0 & \frac{1}{3}ml^2 \end{bmatrix}\right] \begin{Bmatrix} A_x \\ A_\theta \\ A_\phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Setting the determinant of the coefficient matrix to zero leads to the characteristic equation

$$\omega^6 - 18\frac{k}{m}\omega^4 + 96\left(\frac{k}{m}\right)^2\omega^2 - \text{[illegible]}\left(\frac{k}{m}\right)^3 = 0$$

Solution for the roots of this equation yiel

In a similar way, we can determine the mode shapes of the second and third normal modes, given by

$$\begin{Bmatrix} \phi_{x_2} \\ \phi_{\theta_2} \\ \phi_{\phi_2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} \phi_{x_3} \\ \phi_{\theta_3} \\ \phi_{\phi_3} \end{Bmatrix} = \begin{Bmatrix} -\frac{1}{3}(3 - \sqrt{3})l \\ -1 \\ 1 \end{Bmatrix}$$

The general solution for a free vibration of the system may be written as

$$\begin{Bmatrix} x \\ \theta \\ \phi \end{Bmatrix} = \begin{bmatrix} -\frac{1}{3}(3 + \sqrt{3})l & 0 & -\frac{1}{3}(3 - \sqrt{3})l \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

in which

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} C_1 \sin \omega_1 t + C_1' \cos \omega_1 t \\ C_2 \sin \omega_2 t + C_2' \cos \omega_2 t \\ C_3 \sin \omega_3 t + C_3' \cos \omega_3 t \end{Bmatrix}$$

For small displacements in the normal coordinates q_1, q_2, q_3, the configurations of the system are shown in Fig. 5-3. Note that the mode shapes are alternately symmetric and antisymmetric around the plane of symmetry through the pin. The first and third modes, the symmetric modes, each have two nodal points. For the second mode, which is antisymmetric, the nodal point is at the pin.

Suppose the system is set in motion with the displacements $\theta = \theta_0$ and $x = \phi = 0$, the velocities being initially zero. The initial conditions in the normal coordinates are given by the solution of

$$\begin{Bmatrix} 0 \\ \theta_0 \\ 0 \end{Bmatrix} = \begin{bmatrix} -\frac{1}{3}(3 + \sqrt{3})l & 0 & -\frac{1}{3}(3 - \sqrt{3})l \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} -\frac{1}{3}(3 + \sqrt{3})l & 0 & -\frac{1}{3}(3 - \sqrt{3})l \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix}$$

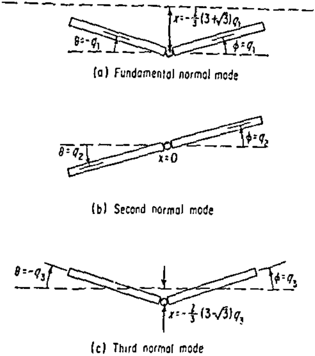


Fig. 5-3 *Normal mode shapes of the system of Fig 5-1.*

Premultiplying by the inverse of the square matrix $[\phi]$, we can write the initial conditions as

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} -\frac{1}{4}(1 - \sqrt{3}) \\ \frac{1}{2} \\ -\frac{1}{4}(1 + \sqrt{3}) \end{Bmatrix} \theta_0$$

$$\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Substitution of the initial conditions into the solution for the free vibrations in the normal coordinates leads to

$$\begin{aligned} q_1 &= -\tfrac{1}{4}(1 - \sqrt{3})\theta_0 \cos \omega_1 t \\ q_2 &= \tfrac{1}{2}\theta_0 \cos \omega_2 t \\ q_3 &= -\tfrac{1}{4}(1 + \sqrt{3})\theta_0 \cos \omega_3 t \end{aligned}$$

Making use of the equations of transformation, we can write the solution in terms of the original coordinates as

$$x = -\frac{\sqrt{3}}{6} l\theta_0 \cos \omega_1 t + \frac{\sqrt{3}}{6} l\theta_0 \cos \omega_3 t$$

$$\theta = \tfrac{1}{4}(1 - \sqrt{3})\theta_0 \cos \omega_1 t + \tfrac{1}{2}\theta_0 \cos \omega_2 t + \tfrac{1}{4}(1 + \sqrt{3})\theta_0 \cos \omega_3 t$$

$$\phi = -\tfrac{1}{4}(1 - \sqrt{3})\theta_0 \cos \omega_1 t + \tfrac{1}{2}\theta_0 \cos \omega_2 t - \tfrac{1}{4}(1 + \sqrt{3})\theta_0 \cos \omega_3 t$$

(*b*) *The Cases of Repeated or Zero Eigenvalues*

The eigenvalues may not all be distinct. Let us assume that the kth eigenvalue ω_k^2 is repeated once or that $\omega_k^2 = \omega_{k+1}^2$. For an eigenvalue which is repeated once, the coefficient matrix of Eq. (5.27) can be shown to be of rank $n - 2$. Suppose the first $n - 2$ rows and columns of the coefficient matrix represents a nonsingular matrix. Then we can solve $n - 2$ of the n equations of Eq. (5.27) for $n - 2$ of the amplitudes in terms of the remaining two amplitudes. We can write

$$\begin{Bmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{n-2,j} \end{Bmatrix} = -\left[\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1,n-2} \\ k_{21} & k_{22} & \cdots & \cdot \\ \vdots & & & \vdots \\ k_{n-2,1} & \cdot & \cdots & k_{n-2,n-2} \end{bmatrix} - \omega_k^2 \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1,n-2} \\ m_{21} & m_{22} & \cdots & \cdot \\ \vdots & & & \vdots \\ m_{n-2,1} & \cdot & \cdots & m_{n-2,n-2} \end{bmatrix}\right]^{-1}$$

$$\left[\begin{bmatrix} k_{1,n-1} & k_{1n} \\ k_{2,n-1} & k_{2n} \\ \vdots & \vdots \\ k_{n-2,n-1} & k_{n-2,n} \end{bmatrix} - \omega_k^2 \begin{bmatrix} m_{1,n-1} & m_{1n} \\ m_{2,n-1} & m_{2n} \\ \vdots & \vdots \\ m_{n-2,n-1} & m_{n-2,n} \end{bmatrix}\right] \begin{Bmatrix} A_{n-1,j} \\ A_{nj} \end{Bmatrix} \tag{5.35}$$

in which j represents k or $k + 1$. If the first $n - 2$ rows and columns do not form a nonsingular matrix, we can always form one from another set of $n - 2$ rows and columns. The two amplitudes $A_{n-1,j}$ and A_{nj} can be chosen arbitrarily. Thus there will be two linearly independent solutions for the matrix. $\{A\}_j$. These solutions can be written as

$$\begin{Bmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{nj} \end{Bmatrix} = \begin{Bmatrix} \phi_{1j} \\ \phi_{2j} \\ \vdots \\ \phi_{nj} \end{Bmatrix} C_j \tag{5.36}$$

for $j = k$ and $k + 1$. The matrix $\{\phi\}_j$ represents the shape of the jth normal mode of vibration. As before, the general solution for the free vibrations of the system is given by Eqs. (5.33) and (5.34). Of course, two of the normal modes will have the common natural frequency ω_k. It should be pointed out here that repeated eigenvalues are not common in real problems.

If a rigid-body motion of a system is possible, the first eigenvalue will be $\omega_1^2 = 0$. For the case in which several rigid-body motions are possible, there will be repeated zero eigenvalues. If the system has one rigid-body degree of freedom, we can solve for $n - 1$ of the amplitudes in terms of the remaining amplitude from Eq. (5.29), which reduces to

$$\begin{Bmatrix} A_{11} \\ A_{21} \\ \vdots \\ A_{n-1,1} \end{Bmatrix} = -\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1,n-1} \\ k_{21} & k_{22} & \cdots & \cdot \\ \vdots & & & \\ k_{n-1,1} & & \cdots & k_{n-1,n-1} \end{bmatrix}^{-1} \begin{Bmatrix} k_{1n} \\ k_{2n} \\ \vdots \\ k_{n-1,n} \end{Bmatrix} A_{n1} \quad (5.37)$$

As before, the normal mode shape $\{\phi\}_1$ may be obtained from Eq (5.37) and a suitable normalizing condition. The solution for the motion in the rigid-body mode is given by

$$\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \vdots \\ \phi_{n1} \end{Bmatrix} (C_1 + C_1' t) \quad (5.38)$$

in which C_1 and C_1' are arbitrary constants to be determined from the initial conditions. Thus the general solution for a free vibration of the system can still be written in the form of Eqs. (5.33) and (5.34), except that the motion of the first normal coordinate is given by

$$q_1 = C_1 + C_1' t \quad (5.39)$$

If the system has more than one rigid-body degree of freedom, the motion of the corresponding normal coordinates will have the form of Eq (5.39) The only common case of repeated eigenvalues is that in which the eigenvalues are zero, occurring in a system having more than one rigid-body degree of freedom. The formal procedure just outlined for determining the shape of a rigid-body motion is not really needed in practice We will always be able to write down the rigid-body mode shapes of a system by inspection.

Example 5.5

An idealized system is composed of three rigid bodies, each of mass m, restrained by two springs of stiffness k, as shown in Fig. 5-4. Each of the masses is free to translate in one dimension as shown. Neglecting friction,

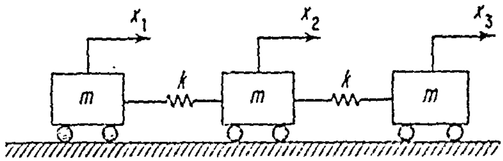


Fig. 5-4

the equations of motion for a free vibration of the system in the coordinates of translation x_1, x_2, x_3 can be written as

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Substitution of the trial solution

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} \sin \omega t$$

into the equations of motion leads to

$$\begin{bmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - m\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Setting the determinant of the coefficient matrix to zero results in the characteristic equation

$$\omega^6 - 4\frac{k}{m}\omega^4 + 3\left(\frac{k}{m}\right)^2\omega^2 = 0$$

which yields the three eigenvalues

$$\omega_1^2 = 0$$
$$\omega_2^2 = \frac{k}{m}$$
$$\omega_3^2 = 3\frac{k}{m}$$

If we substitute the first eigenvalue $\omega_1^2 = 0$ into the algebraic equations in the amplitudes, we can write

$$\begin{bmatrix} k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} A_{11} \\ A_{21} \end{Bmatrix} + \begin{Bmatrix} 0 \\ -k \end{Bmatrix} A_{31} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

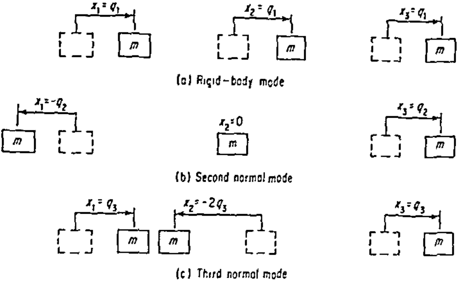


Fig. 5-5 *Normal mode shapes for the system of Fig. 5-4.*

Solution of this equation yields

$$\begin{Bmatrix} A_{11} \\ A_{21} \\ A_{31} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} A_{31}$$

As a normalizing condition, let us arbitrarily set the amplitude of the third mass to unity. Then the rigid-body mode shape is given by

$$\begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

Note that we could have written down this result directly from inspection of the system Obtaining the other two mode shapes, we can write the mode shape matrix as

$$[\phi] = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

The general solution for a free vibration is given by

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

in which

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} C_1 + C_1't \\ C_2 \sin \omega_2 t + C_2' \cos \omega_2 t \\ C_3 \sin \omega_3 t + C_3' \cos \omega_3 t \end{Bmatrix}$$

The configuration of the system displaced in the normal coordinates is shown in Fig. 5-5. Note that the mode shapes are alternately symmetric and antisymmetric around the plane of symmetry at the midpoint of the system.

5.4 Orthogonality of the Normal Modes

(a) *The Orthogonality Relations*

The natural frequencies and normal mode shapes represent solutions of Eq. (5.27). For the ith normal mode, we can write

$$[k]\{\phi\}_i = \omega_i^2[m]\{\phi\}_i \tag{5.40}$$

Similarly, for the jth normal mode

$$[k]\{\phi\}_j = \omega_j^2[m]\{\phi\}_j \tag{5.41}$$

Premultiplying Eqs. (5.40) and (5.41) by the transposes of the jth and ith modal columns, we can write

$$\begin{aligned} \lfloor\phi\rfloor_j[k]\{\phi\}_i &= \omega_i^2\lfloor\phi\rfloor_j[m]\{\phi\}_i \\ \lfloor\phi\rfloor_i[k]\{\phi\}_j &= \omega_j^2\lfloor\phi\rfloor_i[m]\{\phi\}_j \end{aligned} \tag{5.42}$$

Each of the matrix products is equal to its transpose. Since the stiffness matrix is symmetric, the products on the left-hand side of Eqs. (5.42) are the transpose of each other and are equal. Further, since the inertia matrix is symmetric, the matrix products on the right-hand side of these equations are equal. Subtracting the first Eqs. (5.42) from the second, we can write

$$0 = (\omega_j^2 - \omega_i^2)\lfloor\phi\rfloor_i[m]\{\phi\}_j \tag{5.43}$$

If the eigenvalues ω_i^2 and ω_j^2 are unequal, we can conclude from Eqs. (5.42) and (5.43) that

$$\begin{aligned} \lfloor\phi\rfloor_i[m]\{\phi\}_j &= 0 \\ \lfloor\phi\rfloor_i[k]\{\phi\}_j &= 0 \end{aligned} \tag{5.44}$$

For distinct eigenvalues, Eqs. (5.44) require that $i \neq j$. These are statements of the orthogonality of the normal modes. If there are repeated eigenvalues, each of the associated normal modes will be orthogonal to any normal mode associated with a different eigenvalue. Although the normal modes associated with repeated eigenvalues need not be orthogonal to each other, the selection

of mode shapes from Eq. (5.35) is sufficiently arbitrary that they can be required to be orthogonal to each other.

EXAMPLE 5.6

The normal mode shapes for the system of Fig. 5-4 are given in Example 5.5. Let us partially verify the orthogonality of the mode shapes. The statement that the first and second modes are orthogonal to each other with respect to the inertia matrix is given by

$$\lfloor 1\ 1\ 1 \rfloor \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} = 0$$

Similarly, from

$$\lfloor -1\ 0\ 1 \rfloor \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix} = 0$$

it is evident that the second and third modes are orthogonal to each other with respect to the stiffness matrix.

(b) Interpretation of Orthogonality

The orthogonality relations, Eqs. (5.44), can be shown to have the physical meaning that the ith and jth modes are uncoupled, both inertially and elastically. We can write the inertia forces acting on the system as

$$\{F\}_{in} = -[m]\{\ddot{x}\} \tag{5.45}$$

The work done by the inertia forces in a virtual displacement of the system is given by

$$\delta W = \lfloor \delta x \rfloor \{F\}_{in} \tag{5.46}$$

For motion in the jth mode, the accelerations of the system are given by

$$\{\ddot{x}\} = \{\phi\}_j \ddot{q}_j \tag{5.47}$$

For a virtual displacement in the ith normal coordinate, we can write

$$\{\delta x\} = \{\phi\}_i \, \delta q_i \tag{5.48}$$

The work done by the inertia forces resulting from motion in the jth normal mode in a virtual displacement of the ith normal coordinate is given by the combination of Eqs. (5.45) through (5.48), leading to

$$\delta W = -\ddot{q}_j \, \delta q_i \lfloor \phi \rfloor_i [m] \{\phi\}_j \tag{5.49}$$

As a result of the orthogonality relation, the first of Eqs. (5.44), the work done is zero. Thus the inertia forces resulting from motion in one of the normal modes will have no influence on the motion of any other normal mode. We can say that the normal modes are uncoupled inertially. In a similar way, we can show that the normal modes are uncoupled elastically.

Considering the displacement in the ith mode and the inertia force in the jth mode to be vectors, we can interpret orthogonality of the modes as orthogonality of the vectors. The eigenvector $\{\phi\}_i$ represents the displacement of the system per unit displacement of the ith normal coordinate. From Eqs. (5.45) and (5.47), the quantity $-[m]\{\phi\}_j$ represents the inertia forces on the system per unit acceleration of the jth normal coordinate. The product $-\lfloor\phi\rfloor_i[m]\{\phi\}_j$ can be interpreted as the scalar product of a displacement vector and a force vector. Requiring the scalar product to be zero is equivalent to requiring the vectors to be orthogonal. Similar comments can be made with regard to the elastic forces on the system.

(c) *Transformation to the Normal Coordinates*

Since the normal modes are inertially and elastically uncoupled, we can expect the equations of motion in the normal coordinates to be uncoupled. The equations of transformation to the normal coordinates, given by Eq. (5.33), can be written as

$$\{x\} = [\phi]\{q\} \tag{5.50}$$

Substitution of the equation of transformation into the expression for kinetic energy, given by Eq. (5.18), leads to

$$T = \tfrac{1}{2}\lfloor\dot{q}\rfloor[\phi]'[m][\phi]\{\dot{q}\} \tag{5.51}$$

Consider the terms in the matrix product $[\phi]'[m][\phi]$. It can be shown that the term which appears in the ith row and jth column is just $\lfloor\phi\rfloor_i[m]\{\phi\}_j$. From the orthogonality property given by the first of Eqs. (5.44), we can conclude that the terms for which $i \neq j$ will be zero. The matrix product will evidently be diagonal. Let us write

$$M_i = \lfloor\phi\rfloor_i[m]\{\phi\}_i \tag{5.52}$$

in which M_i is the generalized mass of the ith normal mode. The generalized masses will appear on the diagonal of the matrix product. We can rewrite Eq. (5.51) as

$$T = \tfrac{1}{2}\lfloor\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n\rfloor\begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & \cdot \\ \vdots & & & \vdots \\ 0 & \cdot & \cdots & M_n \end{bmatrix}\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{Bmatrix} \tag{5.53}$$

The generalized inertia forces associated with the normal coordinates are given by

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{Bmatrix}_{in} = -\frac{d}{dt}\begin{Bmatrix} \dfrac{\partial T}{\partial \dot{q}_1} \\ \dfrac{\partial T}{\partial \dot{q}_2} \\ \vdots \\ \dfrac{\partial T}{\partial \dot{q}_n} \end{Bmatrix} = -\begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & \cdot \\ \vdots & & & \vdots \\ 0 & \cdot & \cdot\ \cdot & M_n \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_n \end{Bmatrix} \tag{5.54}$$

Substitution of the equations of transformation, Eq. (5.50), into the expression for the elastic potential energy, the first of Eqs. (5.17), leads to

$$U = \tfrac{1}{2}[q][\phi]'[k][\phi]\{q\} \tag{5.55}$$

The term that appears in the ith row and jth column of the matrix product $[\phi]'[k][\phi]$ is represented by $[\phi]_i[k]\{\phi\}_j$. Making use of the orthogonality property given by the second of Eqs. (5.44), it is evident that the off-diagonal terms will be zero. Let us define the generalized stiffness K_i of the ith normal mode by

$$K_i = [\phi]_i[k]\{\phi\}_i \tag{5.56}$$

In view of the definitions, Eqs. (5.52) and (5 56), and of Eqs. (5.42), we can write

$$K_i = M_i\omega_i^2 \tag{5.57}$$

Thus the generalized mass and stiffness have the same relationship to the natural frequency as does the mass and stiffness of a simple mass-spring system The potential energy expression, Eq. (5 55), can be simplified to

$$U = \tfrac{1}{2}[q_1, q_2, \ldots, q_n]\begin{bmatrix} K_1 & 0 & \cdot & 0 \\ 0 & K_2 & & \cdot \\ & & & \\ 0 & & \cdot & K_n \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ \\ q_n \end{Bmatrix} \tag{5.58}$$

Then the generalized *elastic forces associated with the normal coordinates* are given by

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ \cdot \\ Q_n \end{Bmatrix}_{el} = -\begin{Bmatrix} \dfrac{\partial U}{\partial q_1} \\ \dfrac{\partial U}{\partial q_2} \\ \cdot \\ \dfrac{\partial U}{\partial q_n} \end{Bmatrix} = -\begin{bmatrix} K_1 & 0 & \cdot & 0 \\ 0 & K_2 & \cdot & \cdot \\ \cdot & & & \cdot \\ 0 & & \cdot & K_n \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ \cdot \\ q_n \end{Bmatrix} \tag{5.59}$$

For the general case in which external forces are acting on the system, the condition for dynamic equilibrium of the system is given by

$$\{\textstyle\sum Q\} = \{Q\}_{in} + \{Q\}_{el} + \{Q\}_{ex} = \{0\} \tag{5.60}$$

From Eqs. (5.54) and (5.59), the equations of motion can be written in the form

$$\{\textstyle\sum Q\} = -[M]\{\ddot{q}\} - [K]\{q\} + \{Q\}_{ex} = \{0\} \tag{5.61}$$

in which $[M]$ and $[K]$ are the generalized inertia and stiffness matrices. Since these equations are uncoupled, we can write them as

$$\textstyle\sum Q_i = -M_i\ddot{q}_i - K_i q_i + Q_{i,ex} = 0 \tag{5.62}$$

for $i = 1, 2, \ldots, n$.

EXAMPLE 5.7

Let us reexamine the system of Fig. 5-1. From the results of Example 5.4, we can write the equations of transformation to the normal coordinates as

$$\begin{Bmatrix} x \\ \theta \\ \phi \end{Bmatrix} = \begin{bmatrix} -\frac{1}{3}(3+\sqrt{3})l & 0 & -\frac{1}{3}(3-\sqrt{3})l \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

Substitution of the equations of transformation into the expression for the kinetic energy, given in Example 5.2, results in

$$T = \frac{1}{2}\lfloor \dot{q}_1\ \dot{q}_2\ \dot{q}_3 \rfloor \begin{bmatrix} -\frac{1}{3}(3+\sqrt{3})l & -1 & 1 \\ 0 & 1 & 1 \\ -\frac{1}{3}(3-\sqrt{3})l & -1 & 1 \end{bmatrix} \begin{bmatrix} 2m & -\frac{1}{2}ml & \frac{1}{2}ml \\ -\frac{1}{2}ml & \frac{1}{3}ml^2 & 0 \\ \frac{1}{2}ml & 0 & \frac{1}{3}ml^2 \end{bmatrix}$$

$$\times \begin{bmatrix} -\frac{1}{3}(3+\sqrt{3})l & 0 & -\frac{1}{3}(3-\sqrt{3})l \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix}$$

$$= \frac{1}{2}\lfloor \dot{q}_1\ \dot{q}_2\ \dot{q}_3 \rfloor \begin{bmatrix} \frac{4+2\sqrt{3}}{3}ml^2 & 0 & 0 \\ 0 & \frac{2}{3}ml^2 & 0 \\ 0 & 0 & \frac{4-2\sqrt{3}}{3}ml^2 \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix}$$

The fact that the off-diagonal terms are zero represents a useful check on our determination of the normal mode shapes We can write the generalized inertia forces as

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix}_{in} = -\frac{d}{dt}\begin{Bmatrix} \dfrac{\partial T}{\partial \dot{q}_1} \\ \dfrac{\partial T}{\partial \dot{q}_2} \\ \dfrac{\partial T}{\partial \dot{q}_3} \end{Bmatrix} = -\begin{bmatrix} \dfrac{4+2\sqrt{3}}{3}ml^2 & 0 & 0 \\ 0 & \frac{4}{3}ml^2 & 0 \\ 0 & 0 & \dfrac{4-2\sqrt{3}}{3}ml^2 \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix}$$

Making use of the expression for the elastic potential energy given in Example 5.1, we can write

$$U = \tfrac{1}{2}[q_1\, q_2\, q_3]\begin{bmatrix} -\frac{1}{2}(3+\sqrt{3})l & -1 & 1 \\ 0 & 1 & 1 \\ -\frac{1}{2}(3-\sqrt{3})l & -1 & 1 \end{bmatrix}\begin{bmatrix} 6k & -2kl & 2kl \\ -2kl & 2kl^2 & 0 \\ 2kl & 0 & 2kl^2 \end{bmatrix}$$

$$\times\begin{bmatrix} -\frac{1}{2}(3+\sqrt{3})l & 0 & -\frac{1}{2}(3-\sqrt{3})l \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

$$= \tfrac{1}{2}[q_1\, q_2\, q_3]\begin{bmatrix} \frac{4}{3}(3+\sqrt{3})kl^2 & 0 & 0 \\ 0 & 4kl^2 & 0 \\ 0 & 0 & \frac{4}{3}(3-\sqrt{3})kl^2 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_1 \\ q_3 \end{Bmatrix}$$

Then the generalized elastic forces associated with the normal coordinates are

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix}_{el} = -\begin{Bmatrix} \dfrac{\partial U}{\partial q_1} \\ \dfrac{\partial U}{\partial q_2} \\ \dfrac{\partial U}{\partial q_3} \end{Bmatrix} = -\begin{bmatrix} \frac{4}{3}(3+\sqrt{3})kl^2 & 0 & 0 \\ 0 & 4kl^2 & 0 \\ 0 & 0 & \frac{4}{3}(3-\sqrt{3})kl^2 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

Suppose an external force $F(t)$ is applied to the right-hand end of the system as shown in Fig. 5-6 The work done by the applied force in a virtual displacement of the system is given by

$$\delta W = [\delta x\ \delta\theta\ \delta\phi]\begin{Bmatrix} F \\ 0 \\ Fl \end{Bmatrix}$$

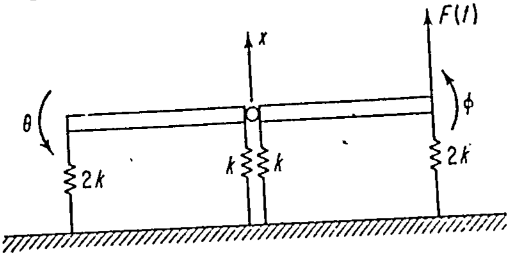


Fig. 5-6

Given the time history of the applied force, we can solve these uncoupled equations separately using the methods of Chap. 3.

EXAMPLE 5.8

A system is made up of two uniform rigid bars, each of mass m and length l, pinned together as shown in Fig. 5-7. Relative angular displacement of the

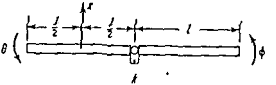


Fig. 5-7

bars is resisted by a torsional spring of stiffness k. Motion of small amplitude can be described by the translation x and the rotations θ and ϕ of the separate bars. The kinetic energy of the system in motion is given by

$$
\begin{aligned}
T &= \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m(\dot{x} + \tfrac{1}{2}l\dot{\theta} + \tfrac{1}{2}l\dot{\phi})^2 + \frac{1}{2}\left(\frac{m}{12}\right)l^2\dot{\theta}^2 + \frac{1}{2}\left(\frac{m}{12}\right)l^2\dot{\phi}^2 \\
&= \tfrac{1}{2}\lfloor \dot{x}\ \dot{\theta}\ \dot{\phi} \rfloor \begin{bmatrix} 2m & \frac{1}{2}ml & \frac{1}{2}ml \\ \frac{1}{2}ml & \frac{1}{3}ml^2 & \frac{1}{4}ml^2 \\ \frac{1}{2}ml & \frac{1}{4}ml^2 & \frac{1}{3}ml^2 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{\theta} \\ \dot{\phi} \end{Bmatrix}
\end{aligned}
$$

We can write the elastic potential energy as

$$
\begin{aligned}
U &= \tfrac{1}{2}k(\phi - \theta)^2 \\
&= \tfrac{1}{2}\lfloor x\ \theta\ \phi \rfloor \begin{bmatrix} 0 & 0 & 0 \\ 0 & k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} x \\ \theta \\ \phi \end{Bmatrix}
\end{aligned}
$$

Then the equations of motion for a free vibration are

$$
\begin{bmatrix} 2m & \frac{1}{2}ml & \frac{1}{2}ml \\ \frac{1}{2}ml & \frac{1}{3}ml^2 & \frac{1}{4}ml^2 \\ \frac{1}{2}ml & \frac{1}{4}ml^2 & \frac{1}{3}ml^2 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} x \\ \theta \\ \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}
$$

Substitution of a trial solution into the equations of motion leads to

$$
\begin{bmatrix} -2m\omega^2 & -\frac{1}{2}ml\omega^2 & -\frac{1}{2}ml\omega^2 \\ -\frac{1}{2}ml\omega^2 & k - \frac{1}{3}ml^2\omega^2 & -k - \frac{1}{4}ml^2\omega^2 \\ -\frac{1}{2}ml\omega^2 & -k - \frac{1}{4}ml^2\omega^2 & k - \frac{1}{3}ml^2\omega^2 \end{bmatrix} \begin{Bmatrix} A_x \\ A_\theta \\ A_\phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}
$$

Solution of the characteristic equation yields the three eigenvalues

$$\omega_1^2 = \omega_2^2 = 0$$
$$\omega_3^2 = 24\frac{k}{ml^2}$$

Substitution of the repeated zero eigenvalue into the algebraic equations in the amplitudes leads to

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} A_x \\ A_\theta \\ A_\phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

which requires that $A_\theta = A_\phi$. Let us arbitrarily select two mode shapes which satisfy this requirement. For example

$$\begin{Bmatrix} \phi_{x_1} \\ \phi_{\theta_1} \\ \phi_{\phi_1} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \phi_{x_2} \\ \phi_{\theta_2} \\ \phi_{\phi_2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$

The first mode shape represents rigid-body translation of the system and the second represents rigid-body rotation around the midpoint of the first bar. The normal mode shape associated with the third eigenvalue is given by

$$\begin{Bmatrix} \phi_{x_3} \\ \phi_{\theta_3} \\ \phi_{\phi_3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1 \\ 1 \end{Bmatrix}$$

in which the rotation ϕ is normalized to unity.

Although the first and second mode shapes must be orthogonal to the third mode shape, they need not be orthogonal to each other. If we write

$$\lfloor 1\ 0\ 0 \rfloor \begin{bmatrix} 2m & \frac{1}{2}ml & \frac{1}{2}ml \\ \frac{1}{2}ml & \frac{1}{3}ml^2 & \frac{1}{4}ml^2 \\ \frac{1}{2}ml & \frac{1}{4}ml^2 & \frac{1}{3}ml^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} = ml$$

we can see that the first and second mode shapes as chosen are not orthogonal with respect to the inertia matrix. Rigid-body modes are never coupled elastically. Thus the first and second mode shapes will be orthogonal with

respect to the stiffness matrix. Let us replace the second normal mode shape by

$$\begin{Bmatrix} \phi_{x_2} \\ \phi_{\theta_2} \\ \phi_{\phi_2} \end{Bmatrix} = \begin{Bmatrix} -\frac{1}{2}l \\ 1 \\ 1 \end{Bmatrix}$$

Then it is evident from

$$\lfloor 1\ 0\ 0 \rfloor \begin{bmatrix} 2m & \frac{1}{2}ml & \frac{1}{2}ml \\ \frac{1}{2}ml & \frac{1}{3}ml^2 & \frac{1}{4}ml^2 \\ \frac{1}{2}ml & \frac{1}{4}ml^2 & \frac{1}{3}ml^2 \end{bmatrix} \begin{Bmatrix} -\frac{1}{2}l \\ 1 \\ 1 \end{Bmatrix} = 0$$

that the rigid-body modes are orthogonal to each other. Note that the second choice for the second normal mode shape represents rigid-body rotation around the midpoint of the system. The most convenient choices for the coordinates needed to describe the rigid-body motion are those that represent translation of the center of mass and rotation around the center of mass. We could of course have written down the appropriate rigid-body mode shapes by inspection

Using the second choice for the second rigid-body mode shape, we can write the equations of transformation to the normal coordinates as

$$\begin{Bmatrix} x \\ \theta \\ \phi \end{Bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2}l & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

If we substitute the equations of transformation into the equations of motion and premultiply the result by the transpose of the mode shape matrix, we arrive at

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2}l & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2m & \frac{1}{2}ml & \frac{1}{2}ml \\ \frac{1}{2}ml & \frac{1}{3}ml^2 & \frac{1}{4}ml^2 \\ \frac{1}{2}ml & \frac{1}{4}ml^2 & \frac{1}{3}ml^2 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2}l & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix}$$
$$+ \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2}l & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2}l & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

which simplifies to

$$\begin{bmatrix} 2m & 0 & 0 \\ 0 & \frac{2}{3}ml^2 & 0 \\ 0 & 0 & \frac{1}{6}ml^2 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4k \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

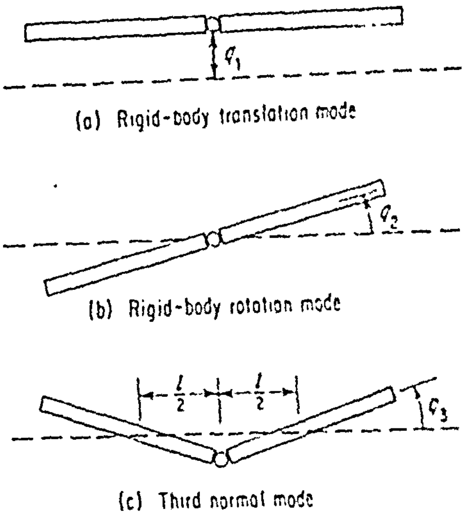


Fig. 5-8 *Normal mode shapes for the system of Fig. 5-7.*

The normal coordinates q_1 and q_2 represent rigid-body translation and rotation around the center of mass. We recognize the inertia terms $2m$ and $\frac{2}{3}ml^2$ as the system mass and moment of inertia taken around the center of mass. The inertia term $\frac{1}{6}ml^2$ and the elastic term $4k$ represent the generalized mass and stiffness of the third normal mode. For small displacements in the normal coordinates, the resulting displacement of the system is shown in Fig. 5-8.

5.5 Effect of Viscous Damping

Consider the addition of viscous damping to the system. For the system in motion, the damping forces associated with the coordinates $x_1, x_2, \ldots, x_n$ will have the form

$$\begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix}_{\text{damp}} = -\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & \cdot \\ \vdots & & & \vdots \\ c_{n1} & \cdot & \cdots & c_{nn} \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{Bmatrix} \tag{5.63}$$

For a motion of small amplitude, we will consider the coefficients c to be constants. Addition of these forces to Eq. (5.24) leads to the general equations for the forced vibrations. The theory for the solution of this set of linear equations, paralleling that given in this chapter for the undamped system, has been given by Foss (Ref. 9).

Suppose we use the normal coordinates determined for the undamped system to describe the motion of the damped system. The work done by the damping forces in a virtual displacement of the system is given by

$$\delta W = [\delta x]\{F\}_{\text{damp}} = -[\delta x][c]\{\dot{x}\} \tag{5.64}$$

Making use of the equations of transformation to the normal coordinates, Eq. (5.33), we can write the work done as

$$\delta W = -[\delta q][\phi]'[c][\phi]\{\dot{q}\} \tag{5.65}$$

Then the generalized damping forces associated with the normal coordinates are

$$\{Q\}_{\text{damp}} = -[\phi]'[c][\phi]\{\dot{q}\} \tag{5.66}$$

in which $[\phi]'[c][\phi]$ represents the transformed matrix of the damping coefficients. As a result of orthogonality, the transformed inertia and stiffness matrices $[\phi]'[m][\phi]$ and $[\phi]'[k][\phi]$ are diagonal. Evidently the transformed damping matrix will be diagonal if $[c]$ can be expressed as a multiple of $[m]$ or $[k]$ or as a linear combination of $[m]$ and $[k]$ Stated differently, we can say that the transformed damping matrix will be diagonal if the damping forces are distributed as the inertia or elastic forces or as a linear combination of these forces In general this is of course not the case and the transformed damping matrix will not be diagonal

A term on the diagonal of $[\phi]'[c][\phi]$ represents the damping of a mode resulting from its own motion An off-diagonal term represents the damping force on a mode resulting from motion of another mode, indicating coupling. In general, the equations of motion in the normal coordinates determined for the undamped system will be coupled by viscous damping For the usual case of small damping it is customary to ignore the off-diagonal coupling terms of the transformed damping matrix. If the damping forces are distributed fairly uniformly through the system it can be argued that the coupling terms will be small. Further, because of the changing phase relationships of the modes with time, the cumulative effect of the coupling terms can be expected to be much smaller than that of the self-damping terms.

The distribution of damping forces through a system cannot ordinarily be determined by analysis The damping level of the system is usually estimated from experience or from experimental measurement. Ordinarily the experimental method yields the damping levels of the pure modes and the terms on the diagonal of the transformed damping matrix. It is not convenient to obtain the off-diagonal terms experimentally.

Ignoring the off-diagonal terms, the generalized damping forces associated with the normal coordinates can be written as

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{Bmatrix}_{\text{damp}} = -\begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & \cdot \\ \vdots & & & \vdots \\ 0 & \cdot & \cdots & c_n \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{Bmatrix} \tag{5.67}$$

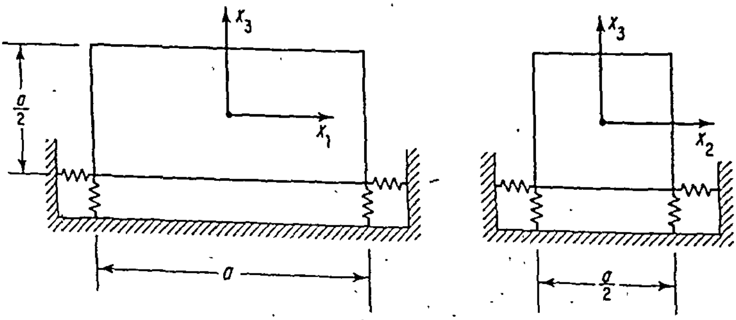
From Eq. (5.62), the resulting uncoupled equations of motion are given by

$$\sum Q_i = -M_i\ddot{q}_i - c_i\dot{q}_i - K_iq_i + Q_{i,ex} = 0 \tag{5.68}$$

Being uncoupled, the equations of motion can be solved using methods outlined for single-degree-of-freedom systems in Chaps. 2 and 3. It should be emphasized that these equations are based on the normal coordinates determined for the undamped system and are approximate in the sense that the off-diagonal terms of the damping matrix have been discarded.

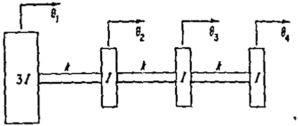
Problems

5-1 A homogeneous block of mass m, having dimensions as shown, rests on springs at its four lower corners. Three springs act on each of the corners. All of the springs are of stiffness k. Let us describe the motion of the block in terms of the translations of the center of mass along the three orthogonal axes x_1, x_2, x_3 and the rotations θ_1, θ_2, θ_3 around these axes. Find the generalized elastic forces using the principle of virtual displacements. Write the equations of motion.



Prob. 5-1

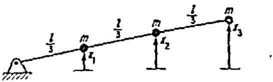
5-2 A uniform shaft free to rotate in bearings carries four equally spaced disks. The axial moment of inertia of the first disk is $3I$ and that of the remaining three disks is I. Derive the generalized elastic forces associated



Prob. 5-2

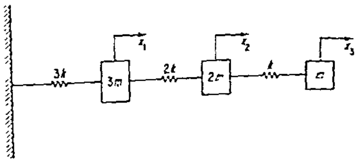
with the coordinates θ_1, θ_2, θ_3, θ_4 making use of the expression for the elastic potential energy. Then write the generalized inertia forces and the equations of motion.

5-3 A lumped-mass representation of a uniform bar pinned at one end is shown. The flexural rigidity of the bar is EI and the masses and dimensions are as shown. The displacements x_1, x_2, x_3 will serve to represent flexural motion of small amplitude of the bar. Find the stiffness influence coefficients associated with x_1, x_2, x_3. Write the generalized elastic and inertia forces and the equations of motion.



Prob. 5-3

5-4 For [illegible] and springs shown, determine the [illegible] matrix for [illegible] potential e[illegible] the equations of motion.

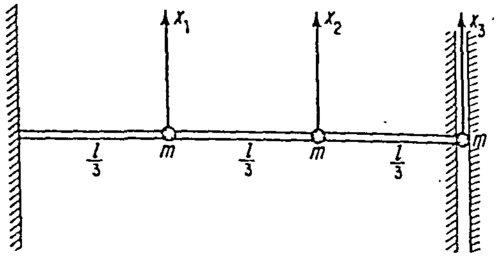


Prob. 5-4

5-5 For a motion of the triple pendulum of small amplitude in the angles $\theta_1, \theta_2, \theta_3$, find the stiffness and flexibility matrices associated with $\theta_1, \theta_2, \theta_3$ making use of the expression for gravitational potential energy. Write the generalized gravitational and inertia forces and the equations of motion.

Prob. 5-5

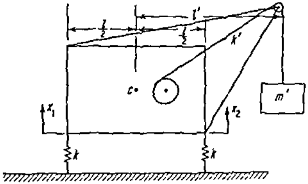
5-6 Three particles of mass m are attached to a massless string which is stretched by a large tensile force T as shown. The right-hand mass is free in the transverse direction. Determine the stiffness and flexibility matrices associated with the transverse displacements x_1, x_2, x_3. Write the equations of motion for transverse vibrations of small amplitude.



Prob. 5-6

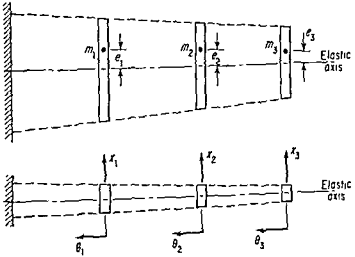
5-7 The center of mass of the hoist of mass m is at C. The moment of inertia of the hoist around C is given by I. The hoist rests on two springs of spring constant k. A mass m' is being supported by the hoist. The coordinates

x_1, x_2 represent vertical motion of the corners of the hoist. Let the coordinate x_3 represent the elongation of the support cable, whose stiffness may be represented by the spring constant k'. Consider a small vertical motion of the system. Find the generalized inertia forces associated with the coordinates x_1, x_2, x_3 making use of the principle of virtual displacements.



Prob. 5-7

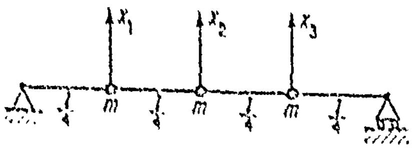
5-8 In a flutter model of an airplane wing, the mass is represented by the three elements of mass m_1, m_2, and m_3 as shown. The motion of the elements is described by the vertical displacements x_1, x_2, and x_3 of the points on the elastic axis and the rotations θ_1, θ_2, and θ_3 around the elastic axis. The



Prob. 5-8

moments of inertia of the mass elements around the elastic axis are represented by I_1, I_2, and I_3. The centers of mass of the mass elements are to the rear of the elastic axis by the distances e_1, e_2, and e_3. Determine the generalized inertia forces associated with the six coordinates making use of the expression for kinetic energy.

5-9 A uniform simply supported beam of flexural rigidity EI carries three equally spaced masses m. Write the equations of motion for the beam in terms of the displacements x_1, x_2, x_3. For a free vibration, obtain the characteristic equation and the natural frequencies. Solve for the normal mode shapes. Note that the mode shapes are symmetric or antisymmetric around the midpoint of the beam.



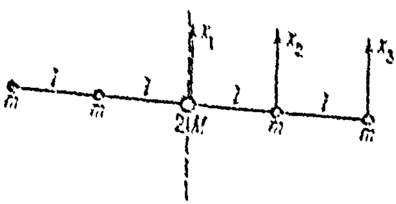
Prob. 5-9

5-10 Equal and opposite torques are applied to the end disks of the system of Prob. 5-2 resulting in the relative rotation $\theta_4 - \theta_1 = \Delta\theta$. The disks are released and the system experiences free vibrations. Write the solution for the free vibrations of the system in terms of the normal modes.

5-11 Determine the natural frequencies and normal mode shapes for the system of Prob. 5-1.

5-12 Determine the natural frequencies and normal mode shapes for the system of Prob. 5-3.

5-13 An airplane wing is idealized as a uniform free-free beam of flexural rigidity EI carrying five masses as shown. The large mass $2M$ represents the fuselage. If we are interested in the symmetric modes of flexural vibration, we need only consider the system on one side of the plane of symmetry. Write the equations of motion for the right-hand half of the system using the coordinates x_1, x_2, x_3. Determine the natural frequencies and normal mode shapes for the symmetric modes if $m/M = 0.1$.

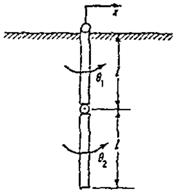


Prob. 5-13

5-14 With the triple pendulum of Prob. 5-5 at equilibrium, the bottom mass is suddenly given a velocity V to the right. Find the resulting free vibrations of the system in terms of the normal modes of vibration.

5-15 Determine the natural frequencies and normal mode shapes for the system of Prob. 5-4.

5-16 Two uniform bars of mass m and length l are pinned together as shown. The upper bar is pinned to a small roller of negligible mass and friction. Find the natural frequencies and normal mode shapes for a free vibration of the system.



Prob 5-16

5-17 Show that the normal mode shapes for the system of Prob. 5-9 are orthogonal with respect to the inertia and stiffness matrices.

5-18 Transform the equations for a symmetric motion of the system of Prob. 5-13 to the normal coordinates

5-19 Write the equations of motion for a free vibration of the system of Prob. 5-1 in the normal coordinates. You will need the results of Prob. 5-11

5-20 Assume that the triple pendulum of Prob 5-5 is immersed in a liquid and that the identical masses experience viscous damping forces. Using the principle of virtual displacements, write the generalized damping forces associated with the angles θ_1, θ_2, θ_3. Note that the damping matrix has the form of the inertia matrix. Thus, damping will not couple the equations of motion in the normal coordinates.

Methods for Determining the Normal Modes of Vibration of Lumped-Mass Systems

6.1 Solution by the Method of Iteration

(a) The Basic Scheme of Iteration

The classical method outlined in Sec. 5.3 for determining the natural frequencies and normal mode shapes is particularly suitable if the system has only a few degrees of freedom. For systems having more than a few degrees of freedom, the operations required may involve considerable labor and other methods of solution should be considered. In particular, if the characteristics of only a few of the normal modes are needed, an iterative method may be preferred. In real problems, it is often the case that only a few of the lower modes of vibration are important.

In this section we will consider an iterative method for solving Eq. (5.27) which is initiated with a trial eigenvector. If we premultiply Eq. (5.27) by the inverse of the stiffness matrix and rearrange the result, we can write

$$\frac{1}{\omega^2}\{A\} = [k]^{-1}[m]\{A\} \tag{6.1}$$

Let us select a trial eigenvector $\{A\}^{(0)}$, arbitrarily setting $A_n^{(0)} = 1$ as a normalizing condition. Substitution of the trial eigenvector into the right-hand side of Eq. (6.1) leads to

$$\{N\}^{(1)} = [k]^{-1}[m]\{A\}^{(0)} \tag{6.2}$$

We can normalize the column matrix $\{N\}^{(1)}$ with the nth element equal to

unity by factoring out $N_n^{(1)}$. Comparing this result with the left-hand side of Eq. (6.1), we can write

$$\{A\}^{(1)} = \begin{Bmatrix} \dfrac{N_1}{N_n} \\ \dfrac{N_2}{N_n} \\ \vdots \\ 1 \end{Bmatrix}^{(1)} \qquad \frac{1}{\omega^2_{(1)}} = N_n^{(1)} \tag{6.3}$$

in which $\{A\}^{(1)}$ is an improved trial eigenvector and $\omega^2_{(1)}$ is a first estimate of the associated eigenvalue. In the event that $\{A\}^{(0)}$ represents one of the exact eigenvectors of the system, the result $\{A\}^{(1)}$ will be identical and $\omega^2_{(1)}$ will be the exact eigenvalue. Otherwise $\{A\}^{(1)}$ will differ from $\{A\}^{(0)}$. The iterative process is continued by the substitution of the improved trial eigenvector into the right-hand side of Eq (6 1), leading to

$$\{N\}^{(2)} = [k]^{-1}[m]\{A\}^{(1)} \tag{6.4}$$

and to

$$\{A\}^{(2)} = \begin{Bmatrix} \dfrac{N_1}{N_n} \\ \dfrac{N_2}{N_n} \\ \vdots \\ 1 \end{Bmatrix}^{(2)} \qquad \frac{1}{\omega^2_{(2)}} = N_n^{(2)} \tag{6.5}$$

in which $\{A\}^{(2)}$ is a further improved trial eigenvector and $\omega^2_{(2)}$ is a second estimate of the eigenvalue The procedure outlined can be repeated as many times as are required It will be shown that the results converge on the fundamental eigenvector $\{\phi\}_1$ and the eigenvalue ω_1^2

The n eigenvectors of the system represent all possible configurations of the system. Thus we can express the original trial eigenvector as a linear combination of the eigenvectors of the system, as given by

$$\{A\}^{(0)} = [\phi]\{c\} \tag{6 6}$$

in which the columns of $[\phi]$ are the system eigenvectors The coefficients $\{c\}$ represent the amplitudes of the eigenvectors Substitution of Eq. (6.6) into the right-hand side of Eq (6.1) results in

$$\{N\}^{(1)} = [k]^{-1}[m][\phi]\{c\} \tag{6 7}$$

The ith eigenvector and eigenvalue must satisfy Eq. (6.1), as given by

$$\frac{1}{\omega_i^2}\{\phi\}_i = [k]^{-1}[m]\{\phi\}_i \tag{6.8}$$

Combining the expressions of the form of Eq. (6.8) for all n normal modes, we can write

$$[\phi]\left[\diagdown\frac{1}{\omega^2}\diagdown\right] = [k]^{-1}[m][\phi] \tag{6.9}$$

in which

$$\left[\diagdown\frac{1}{\omega^2}\diagdown\right] = \begin{bmatrix} \frac{1}{\omega_1^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{\omega_2^2} & \cdots & \cdot \\ \vdots & & & \vdots \\ 0 & \cdot & \cdots & \frac{1}{\omega_n^2} \end{bmatrix} \tag{6.10}$$

Substitution of Eq. (6.9) into Eq. (6.7) leads to

$$\{N\}^{(1)} = [\phi]\left[\diagdown\frac{1}{\omega^2}\diagdown\right]\{c\} \tag{6.11}$$

Then the improved trial eigenvector is given by

$$\{A\}^{(1)} = \frac{1}{N_n^{(1)}}[\phi]\left[\diagdown\frac{1}{\omega^2}\diagdown\right]\{c\} \tag{6.12}$$

After r iterations, we can write

$$\{A\}^{(r)} = \frac{1}{N_n^{(1)}\cdots N_n^{(r)}}[\phi]\left[\diagdown\frac{1}{\omega^{2r}}\diagdown\right]\{c\} \tag{6.13}$$

in which

$$\left[\diagdown\frac{1}{\omega^{2r}}\diagdown\right] = \begin{bmatrix} \frac{1}{\omega_1^{2r}} & 0 & \cdots & 0 \\ 0 & \frac{1}{\omega_2^{2r}} & \cdots & \cdot \\ \vdots & & & \vdots \\ 0 & \cdot & \cdots & \frac{1}{\omega_n^{2r}} \end{bmatrix} \tag{6.14}$$

Since $\omega_1 < \omega_2 < \cdots < \omega_n$, the first term on the diagonal of the matrix of Eq. (6.14) becomes the dominant term after several iterations. Then we can approximate Eq. (6.13) by

$$\{A\}^{(r)} = \frac{1}{N_n^{(1)}\cdots N_n^{(r)}}\{\phi\}_1\frac{1}{\omega_1^{2r}}c_1 \tag{6.15}$$

provided that $c_1 \neq 0$. It is evident that the trial eigenvector converges on the fundamental eigenvector $\{\phi\}_1$ of the system. Further, the fundamental eigenvalue is approximated more and more closely by

$$\frac{1}{\omega_1^2} = N_n^{(r)} \tag{6.16}$$

For an original trial eigenvector which is a close estimate of the fundamental eigenvector, we can expect c_1 to be substantially larger than the other coefficients. If this is the case, the convergence of the trial eigenvector to the form of Eq. (6.15) will be more rapid. The rate of convergence of the trial eigenvector will also be enhanced if the eigenvalues of the system are well spaced. For example, we can expect slow convergence if the second eigenvalue is only a little larger than the fundamental eigenvalue.

If we premultiply Eq. (5.27) by the inverse of the inertia matrix and rearrange the result, we can write

$$\omega^2\{A\} = [m]^{-1}[k]\{A\} \tag{6.17}$$

which represents an alternate form of the equations to that of Eq. (6.1). If the iterative procedure described earlier is followed using Eq. (6.17), it can be shown that the trial eigenvector will converge on the highest eigenvector $\{\phi\}_n$. The procedure, of course, also produces the highest eigenvalue ω_n^2. In most cases, it is the lower eigenvalues and eigenvectors which are needed and the form of the equations represented by Eq. (6.1) is most useful. For further information, consult Ref. 10.

Example 6.1

Consider the system illustrated by Fig. 5-1. Making use of equations developed in Example 5.4, we can write a set of equations having the form of Eq. (6.1), given by

$$\frac{1}{\omega^2}\begin{Bmatrix} A_x \\ A_\theta \\ A_\phi \end{Bmatrix} = \begin{bmatrix} 6k & -2kl & 2kl \\ -2kl & 2kl^2 & 0 \\ 2kl & 0 & 2kl^2 \end{bmatrix}^{-1} \begin{bmatrix} 2m & -\frac{1}{2}ml & \frac{1}{2}ml \\ -\frac{1}{2}ml & \frac{1}{3}ml^2 & 0 \\ \frac{1}{2}ml & 0 & \frac{1}{3}ml^2 \end{bmatrix} \begin{Bmatrix} A_x \\ A_\theta \\ A_\phi \end{Bmatrix}$$

Performing the matrix operations, we can simplify the equations to

$$\frac{1}{\omega^2}\begin{Bmatrix} A_x \\ A_\theta \\ A_\phi \end{Bmatrix} = \frac{m}{12k}\begin{bmatrix} 6 & -l & l \\ \frac{3}{l} & 1 & 1 \\ -\frac{3}{l} & 1 & 1 \end{bmatrix} \begin{Bmatrix} A_x \\ A_\theta \\ A_\phi \end{Bmatrix}$$

Assuming that we are unable to make a rational guess at the shape of the fundamental mode, let us arbitrarily try

$$\{A\}^{(0)} = \begin{Bmatrix} l \\ 1 \\ 1 \end{Bmatrix}.$$

in which l is the bar length. Note that th[illegible]ensions of the elements of the trial eigenvector must correspond to [illegible] ns [illegible] oordinates, in this case a length and two angles. [illegible] vector into the right-hand side of the equatio[illegible]

(b) Extension of the Method to Higher Modes

We have seen that the fundamental eigenvector and eigenvalue can be obtained by an iterative procedure using Eq. (6.1). The procedure can be extended to produce the second eigenvector and eigenvalue by making use of the property of orthogonality of the normal modes. From the first of the orthogonality relations, Eqs. (5.44), a trial eigenvector $\{A\}^{(0)}$ will be orthogonal to the fundamental eigenvector $\{\phi\}_1$ with respect to the inertia matrix if

$$\lfloor\phi\rfloor_1[m]\{A\}^{(0)} = 0 \tag{6.18}$$

If we begin with a trial eigenvector which satisfies Eq. (6.18), the iterative procedure will converge on the second eigenvector and eigenvalue of the system.

To demonstrate the validity of the proposed method, let us premultiply Eq. (6.6) by the inertia matrix and then by the row matrix representing the fundamental eigenvector, leading to

$$\lfloor\phi\rfloor_1[m]\{A\}^{(0)} = \lfloor\phi\rfloor_1[m][\phi]\{c\} \tag{6.19}$$

In view of the first of the orthogonality relations, Eqs. (5.44), and of the definition of the generalized masses, Eq. (5.52), we can simplify the right-hand side of this equation to

$$\lfloor\phi\rfloor_1[m]\{A\}^{(0)} = M_1 c_1 \tag{6.20}$$

in which M_1 is the generalized mass of the fundamental mode. Thus if we require the trial eigenvector to satisfy Eq. (6.18), then $c_1 = 0$. We can say that the trial eigenvector contains none of the fundamental eigenvector Then the improved trial eigenvector after r iterations, given by Eq. (6.13), can be approximated by

$$\{A\}^{(r)} = \frac{1}{N_n^{(1)}, \ldots, N_n^{(r)}}\{\phi\}_2\frac{1}{\omega_2^{2r}}c_2 \tag{6.21}$$

provided that $c_2 \neq 0$ Evidently the trial eigenvector will converge on the second eigenvector $\{\phi\}_2$ of the system Further, the second eigenvalue will be approximated more and more closely by

$$\frac{1}{\omega_2^2} \approx N_n^{(r)} \tag{6.22}$$

Let us expand Eq. (6.18) and solve for the amplitude A_1 resulting in

$$A_1 = S_{12}A_2 + S_{13}A_3 + \cdots + S_{1n}A_n \tag{6.23}$$

in which the quantities S are coefficients. Consider the relation

$$\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ \cdot \\ A_n \end{Bmatrix} = \begin{bmatrix} 0, & S_{12} & S_{13} & & S_{1n} \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \cdot & & \vdots & & \cdot \\ 0 & 0 & 0 & \cdot & 1 \end{bmatrix} \begin{Bmatrix} A_1^* \\ A_2 \\ A_3 \\ \cdot \\ A_n \end{Bmatrix} \tag{6.24}$$

eigenvalue. From the results of Example 6.1, a trial eigenvector will be orthogonal to the fundamental eigenvector with respect to the inertia matrix if

$$\lfloor -1.53l, -1.00, 1\rfloor \begin{bmatrix} 2m & -\frac{1}{2}ml & \frac{1}{2}ml \\ -\frac{1}{2}ml & \frac{1}{3}ml^2 & 0 \\ \frac{1}{2}ml & 0 & \frac{1}{3}ml^2 \end{bmatrix} \begin{Bmatrix} A_x \\ A_\theta \\ A_\phi \end{Bmatrix} = 0$$

Performing the multiplications, we can write

$$A_x = 0.212lA_\theta - 0.212lA_\phi$$

Then the sweeping matrix is

$$[S]_1 = \begin{bmatrix} 0 & 0.212l & -0.212l \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The set of equations which forms the basis for the iterative procedure, given in Example 6.1, can be modified to

$$\frac{1}{\omega^2}\begin{Bmatrix} A_x \\ A_\theta \\ A_\phi \end{Bmatrix} = \frac{m}{12k}\begin{bmatrix} 6 & -l & l \\ \frac{3}{l} & 1 & 1 \\ -\frac{3}{l} & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0.212l & -0.212l \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} A_x \\ A_\theta \\ A_\phi \end{Bmatrix}$$

$$= \frac{m}{12k}\begin{bmatrix} 0 & 0.272l & -0.272l \\ 0 & 1.64 & 0.364 \\ 0 & 0.364 & 1.64 \end{bmatrix}\begin{Bmatrix} A_x \\ A_\theta \\ A_\phi \end{Bmatrix}$$

After several iterations, we will arrive at the result

$$\{N\} = 0.167\frac{m}{k}\begin{Bmatrix} 0.00 \\ 1.00 \\ 1 \end{Bmatrix}$$

Then the second eigenvector and eigenvalue of the system are

$$\{\phi\} = \begin{Bmatrix} 0.00 \\ 1.00 \\ 1 \end{Bmatrix}$$

$$\omega_2^2 = \frac{k}{0.167m} = 6.00\frac{k}{m}$$

which compare with the results of Example 5.4.

(c) *An Alternate Approach for the Higher Modes; Systems with Rigid-Body Degrees of Freedom*

Let us consider an alternate method for obtaining the higher eigenvectors and eigenvalues which does not involve the insertion of sweeping matrices.

If an arbitrary trial eigenvector is substituted into the right-hand side of this equation, the result is a new trial eigenvector which satisfies Eq. (6.18). Note that the only difference in $\{A\}$ on the two sides of this equation lies in the first term. We can rewrite Eq. (6.1) as

$$\frac{1}{\omega^2}\{A\} = [k]^{-1}[m][S]_1\{A\} \tag{6.25}$$

in which $[S]_1$ referred to as the sweeping matrix, is just the square matrix of Eq. (6.24). From the iterative procedure, we expect the trial eigenvector to converge on the second eigenvector of the system.

If we would like to obtain a third eigenvector and eigenvalue, we can follow a similar procedure. A trial eigenvector will be orthogonal to the first two eigenvectors with respect to the inertia matrix if

$$\begin{bmatrix} \lfloor\phi\rfloor_1 \\ \lfloor\phi\rfloor_2 \end{bmatrix} [m]\{A\}^{(0)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{6.26}$$

Solving for two of the amplitudes, arbitrarily A_1 and A_2, in terms of the remaining amplitudes, we can write

$$\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{bmatrix} S_{13} & S_{14} & \cdots & S_{1n} \\ S_{23} & S_{24} & \cdots & S_{2n} \end{bmatrix} \begin{Bmatrix} A_3 \\ A_4 \\ \vdots \\ A_n \end{Bmatrix} \tag{6.27}$$

According to the relation

$$\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ \vdots \\ A_n \end{Bmatrix} = \begin{bmatrix} 0 & 0 & S_{13} & S_{14} & \cdots & S_{1n} \\ 0 & 0 & S_{23} & S_{24} & \cdots & S_{2n} \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{Bmatrix} A_1^* \\ A_2^* \\ A_3 \\ A_4 \\ \vdots \\ A_n \end{Bmatrix} \tag{6.28}$$

a trial eigenvector substituted into the right-hand side is transformed into a new trial eigenvector which satisfies Eq. (6.26). We can rewrite Eq. (6.1) as

$$\frac{1}{\omega^2}\{A\} = [k]^{-1}[m][S]_2\{A\} \tag{6.29}$$

in which the sweeping matrix $[S]_2$ is the square matrix of Eq. (6.28). Using the iterative procedure, we will find that the trial eigenvector converges on the third eigenvector of the system.

EXAMPLE *6.2*

Having obtained in Example 6.1 the fundamental eigenvector and eigenvalue of the system of Fig. 5-1, let us determine the second eigenvector and

eigenvalue. From the results of Example 6.1, a trial eigenvector will be orthogonal to the fundamental eigenvector with respect to the inertia matrix if

$$[-1.53l, -1.00, 1]\begin{bmatrix} 2m & -\frac{1}{2}ml & \frac{1}{2}ml \\ -\frac{1}{2}ml & \frac{1}{3}ml^2 & 0 \\ \frac{1}{2}ml & 0 & \frac{1}{3}ml^2 \end{bmatrix}\begin{Bmatrix} A_x \\ A_\theta \\ A_\phi \end{Bmatrix} = 0$$

Performing the multiplications, we can write

$$A_x = 0.212lA_\theta - 0.212lA_\phi$$

Then the sweeping matrix is

$$[S]_1 = \begin{bmatrix} 0 & 0.212l & -0.212l \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The set of equations which forms the basis for the iterative procedure, given in Example 6.1, can be modified to

$$\frac{1}{\omega^2}\begin{Bmatrix} A_x \\ A_\theta \\ A_\phi \end{Bmatrix} = \frac{m}{12k}\begin{bmatrix} 6 & -l & l \\ \frac{3}{l} & 1 & 1 \\ -\frac{3}{l} & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0.212l & -0.212l \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} A_x \\ A_\theta \\ A_\phi \end{Bmatrix}$$

$$= \frac{m}{12k}\begin{bmatrix} 0 & 0.272l & -0.272l \\ 0 & 1.64 & 0.364 \\ 0 & 0.364 & 1.64 \end{bmatrix}\begin{Bmatrix} A_x \\ A_\theta \\ A_\phi \end{Bmatrix}$$

After several iterations, we will arrive at the result

$$\{N\} = 0.167\frac{m}{k}\begin{Bmatrix} 0.00 \\ 1.00 \\ 1 \end{Bmatrix}$$

Then the second eigenvector and eigenvalue of the system are

$$\{\phi\} = \begin{Bmatrix} 0.00 \\ 1.00 \\ 1 \end{Bmatrix}$$

$$\omega_2^2 = \frac{k}{0.167m} = 6.00\frac{k}{m}$$

which compare with the results of Example 5.4.

(c) *An Alternate Approach for the Higher Modes, Systems with Rigid-Body Degrees of Freedom*

Let us consider an alternate method for obtaining the higher eigenvectors and eigenvalues which does not involve the insertion of sweeping matrices

If an arbitrary trial eigenvector is substituted into the right-hand side of this equation, the result is a new trial eigenvector which satisfies Eq. (6.18). Note that the only difference in $\{A\}$ on the two sides of this equation lies in the first term. We can rewrite Eq. (6.1) as

$$\frac{1}{\omega^2}\{A\} = [k]^{-1}[m][S]_1\{A\} \tag{6.25}$$

in which $[S]_1$ referred to as the sweeping matrix, is just the square matrix of Eq. (6.24). From the iterative procedure, we expect the trial eigenvector to converge on the second eigenvector of the system.

If we would like to obtain a third eigenvector and eigenvalue, we can follow a similar procedure. A trial eigenvector will be orthogonal to the first two eigenvectors with respect to the inertia matrix if

$$\begin{bmatrix} \lfloor\phi\rfloor_1 \\ \lfloor\phi\rfloor_2 \end{bmatrix}[m]\{A\}^{(0)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{6.26}$$

Solving for two of the amplitudes, arbitrarily A_1 and A_2, in terms of the remaining amplitudes, we can write

$$\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{bmatrix} S_{13} & S_{14} & \cdots & S_{1n} \\ S_{23} & S_{24} & \cdots & S_{2n} \end{bmatrix}\begin{Bmatrix} A_3 \\ A_4 \\ \vdots \\ A_n \end{Bmatrix} \tag{6.27}$$

According to the relation

$$\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ \vdots \\ A_n \end{Bmatrix} = \begin{bmatrix} 0 & 0 & S_{13} & S_{14} & \cdots & S_{1n} \\ 0 & 0 & S_{23} & S_{24} & \cdots & S_{2n} \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}\begin{Bmatrix} A_1^* \\ A_2^* \\ A_3 \\ A_4 \\ \vdots \\ A_n \end{Bmatrix} \tag{6.28}$$

a trial eigenvector substituted into the right-hand side is transformed into a new trial eigenvector which satisfies Eq. (6.26). We can rewrite Eq. (6.1) as

$$\frac{1}{\omega^2}\{A\} = [k]^{-1}[m][S]_2\{A\} \tag{6.29}$$

in which the sweeping matrix $[S]_2$ is the square matrix of Eq. (6.28). Using the iterative procedure, we will find that the trial eigenvector converges on the third eigenvector of the system.

EXAMPLE 6.2

Having obtained in Example 6.1 the fundamental eigenvector and eigenvalue of the system of Fig. 5-1, let us determine the second eigenvector and

eigenvalue. From the results of Example 6.1, [illegible] orthogonal to the fundamental eigenvector with respect to the [illegible]

$$[-1.57l \quad -1.00 \quad 1]\begin{bmatrix} 2m & -\frac{1}{2}ml & \frac{1}{2}ml \\ -\frac{1}{2}ml & \frac{1}{2}ml^2 & 0 \\ \frac{1}{2}ml & 0 & \frac{1}{2}ml^2 \end{bmatrix}\begin{Bmatrix} \delta_a \\ \theta_b \\ \theta_c \end{Bmatrix} = 0$$

Performing the multiplications, we can write

$$\delta_a = 0.212l\theta_b - 0.212l\theta_c$$

Then the sweeping matrix is

$$[S]_1 = \begin{bmatrix} 0 & 0.212l & -0.212l \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The set of equations which forms the basis for the iterative procedure, given in Example 6.1, can be modified to

$$\frac{1}{\omega^2}\begin{Bmatrix} \delta_a \\ \theta_b \\ \theta_c \end{Bmatrix} = \frac{m}{12k}\begin{bmatrix} 6 & l & l \\ \frac{3}{l} & 1 & 1 \\ \frac{3}{l} & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0.212l & 0.212l \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \delta_a \\ \theta_b \\ \theta_c \end{Bmatrix}$$

$$= \frac{m}{12k}\begin{bmatrix} 0 & 0.272l & 0.272l \\ 0 & 1.64 & 0.364 \\ 0 & 0.364 & 1.64 \end{bmatrix}\begin{Bmatrix} \delta_a \\ \theta_b \\ \theta_c \end{Bmatrix}$$

After several iterations, we will arrive at the result

$$[illegible] = 0.167\frac{m}{k}\begin{Bmatrix} 0.00 \\ 1.00 \\ [illegible] \end{Bmatrix}$$

Then the second eigenvector and eigenvalue of the system are

$$\{\phi\} = \begin{Bmatrix} 0.00 \\ 1.00 \\ [illegible] \end{Bmatrix}$$

$$\omega_2^2 = \frac{k}{0.167m} = 6.00\frac{k}{m}$$

which compare with the results of Example 5.4.

(c) An Alternate Approach [illegible] Degrees of Freedom

Let us consider an alternate [illegible] and eigenvalues which [illegible]

If an arbitrary trial eigenvector is substituted into the right-hand side of this equation, the result is a new trial eigenvector which satisfies Eq. (6.18). Note that the only difference in $\{A\}$ on the two sides of this equation lies in the first term. We can rewrite Eq. (6.1) as

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in which $[S]_1$ referred to as the sweeping matrix, is just the square matrix of Eq. (6.24). From the iterative procedure, we expect the trial eigenvector to converge on the second eigenvector of the system.

If we would like to obtain a third eigenvector and eigenvalue, we can follow a similar procedure. A trial eigenvector will be orthogonal to the first two eigenvectors with respect to the inertia matrix if

$$\begin{bmatrix} \lfloor\phi\rfloor_1 \\ \lfloor\phi\rfloor_2 \end{bmatrix}[m]\{A\}^{(0)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{6.26}$$

Solving for two of the amplitudes, arbitrarily A_1 and A_2, in terms of the remaining amplitudes, we can write

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According to the relation

$$\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ \vdots \\ A_n \end{Bmatrix} = \begin{bmatrix} 0 & 0 & S_{13} & S_{14} & \cdots & S_{1n} \\ 0 & 0 & S_{23} & S_{24} & \cdots & S_{2n} \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}\begin{Bmatrix} A_1^* \\ A_2^* \\ A_3 \\ A_4 \\ \vdots \\ A_n \end{Bmatrix} \tag{6.28}$$

a trial eigenvector substituted into the right-hand side is transformed into a new trial eigenvector which satisfies Eq. (6.26). We can rewrite Eq. (6.1) as

$$\frac{1}{\omega^2}\{A\} = [k]^{-1}[m][S]_2\{A\} \tag{6.29}$$

in which the sweeping matrix $[S]_2$ is the square matrix of Eq. (6.28). Using the iterative procedure, we will find that the trial eigenvector converges on the third eigenvector of the system.

EXAMPLE 6.2

Having obtained in Example 6.1 the fundamental eigenvector and eigenvalue of the system of Fig. 5-1, let us determine the second eigenvector and

eigenvalue. From the results of Example 6.1, a trial eigenvector will be orthogonal to the fundamental eigenvector with respect to the inertia matrix if

$$\lfloor -1.53l, -1.00, 1 \rfloor \begin{bmatrix} 2m & -\frac{1}{2}ml & \frac{1}{2}ml \\ -\frac{1}{2}ml & \frac{1}{3}ml^2 & 0 \\ \frac{1}{2}ml & 0 & \frac{1}{3}ml^2 \end{bmatrix} \begin{Bmatrix} A_x \\ A_\theta \\ A_\phi \end{Bmatrix} = 0$$

Performing the multiplications, we can write

$$A_x = 0.212lA_\theta - 0.212lA_\phi$$

Then the sweeping matrix is

$$[S]_1 = \begin{bmatrix} 0 & 0.212l & -0.212l \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The set of equations which forms the basis for the iterative procedure, given in Example 6.1, can be modified to

$$\frac{1}{\omega^2}\begin{Bmatrix} A_x \\ A_\theta \\ A_\phi \end{Bmatrix} = \frac{m}{12k}\begin{bmatrix} 6 & -l & l \\ \frac{3}{l} & 1 & 1 \\ -\frac{3}{l} & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0.212l & -0.212l \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} A_x \\ A_\theta \\ A_\phi \end{Bmatrix}$$

$$= \frac{m}{12k}\begin{bmatrix} 0 & 0.272l & -0.272l \\ 0 & 1.64 & 0.364 \\ 0 & 0.364 & 1.64 \end{bmatrix}\begin{Bmatrix} A_x \\ A_\theta \\ A_\phi \end{Bmatrix}$$

After several iterations, we will arrive at the result

$$\{N\} = 0.167\frac{m}{k}\begin{Bmatrix} 0.00 \\ 1.00 \\ 1 \end{Bmatrix}$$

Then the second eigenvector and eigenvalue of the system are

$$\{\phi\} = \begin{Bmatrix} 0.00 \\ 1.00 \\ 1 \end{Bmatrix}$$

$$\omega_2^2 = \frac{k}{0.167m} = 6.00\frac{k}{m}$$

which compare with the results of Example 5.4.

(c) *An Alternate Approach for the Higher Modes: Systems with Rigid-Body Degrees of Freedom*

Let us consider an alternate method for obtaining the higher eigenvectors and eigenvalues which does not involve the insertion of sweeping matrices.

If an arbitrary trial eigenvector is substituted into the right-hand side of this equation, the result is a new trial eigenvector which satisfies Eq. (6.18). Note that the only difference in $\{A\}$ on the two sides of this equation lies in the first term. We can rewrite Eq. (6.1) as

$$\frac{1}{\omega^2}\{A\} = [k]^{-1}[m][S]_1\{A\} \tag{6.25}$$

in which $[S]_1$ referred to as the sweeping matrix, is just the square matrix of Eq. (6.24). From the iterative procedure, we expect the trial eigenvector to converge on the second eigenvector of the system.

If we would like to obtain a third eigenvector and eigenvalue, we can follow a similar procedure. A trial eigenvector will be orthogonal to the first two eigenvectors with respect to the inertia matrix if

$$\begin{bmatrix} \lfloor\phi\rfloor_1 \\ \lfloor\phi\rfloor_2 \end{bmatrix}[m]\{A\}^{(0)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{6.26}$$

Solving for two of the amplitudes, arbitrarily A_1 and A_2, in terms of the remaining amplitudes, we can write

$$\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{bmatrix} S_{13} & S_{14} & \cdots & S_{1n} \\ S_{23} & S_{24} & \cdots & S_{2n} \end{bmatrix} \begin{Bmatrix} A_3 \\ A_4 \\ \vdots \\ A_n \end{Bmatrix} \tag{6.27}$$

According to the relation

$$\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ \vdots \\ A_n \end{Bmatrix} = \begin{bmatrix} 0 & 0 & S_{13} & S_{14} & \cdots & S_{1n} \\ 0 & 0 & S_{23} & S_{24} & \cdots & S_{2n} \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{Bmatrix} A_1^* \\ A_2^* \\ A_3 \\ A_4 \\ \vdots \\ A_n \end{Bmatrix} \tag{6.28}$$

a trial eigenvector substituted into the right-hand side is transformed into a new trial eigenvector which satisfies Eq. (6.26). We can rewrite Eq. (6.1) as

$$\frac{1}{\omega^2}\{A\} = [k]^{-1}[m][S]_2\{A\} \tag{6.29}$$

in which the sweeping matrix $[S]_2$ is the square matrix of Eq. (6.28). Using the iterative procedure, we will find that the trial eigenvector converges on the third eigenvector of the system.

EXAMPLE 6.2

Having obtained in Example 6.1 the fundamental eigenvector and eigenvalue of the system of Fig. 5-1, let us determine the second eigenvector and

eigenvalue. From the results of Example 6.1, a trial eigenvector will be orthogonal to the fundamental eigenvector with respect to the inertia matrix if

$$[-1.53l, -1.00, l]\begin{bmatrix} 2m & -\frac{1}{2}ml & \frac{1}{2}ml \\ -\frac{1}{2}ml & \frac{1}{3}ml^2 & 0 \\ \frac{1}{2}ml & 0 & \frac{1}{3}ml^2 \end{bmatrix}\begin{Bmatrix} A_x \\ A_\theta \\ A_\phi \end{Bmatrix} = 0$$

Performing the multiplications, we can write

$$A_x = 0.212lA_\theta - 0.212lA_\phi$$

Then the sweeping matrix is

$$[S]_1 = \begin{bmatrix} 0 & 0.212l & -0.212l \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The set of equations which forms the basis for the iterative procedure, given in Example 6 1, can be modified to

$$\frac{1}{\omega^2}\begin{Bmatrix} A_x \\ A_\theta \\ A_\phi \end{Bmatrix} = \frac{m}{12k}\begin{bmatrix} 6 & -l & l \\ \frac{3}{l} & 1 & 1 \\ -\frac{3}{l} & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0.212l & -0.212l \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} A_x \\ A_\theta \\ A_\phi \end{Bmatrix}$$

$$= \frac{m}{12k}\begin{bmatrix} 0 & 0.272l & -0.272l \\ 0 & 1.64 & 0.364 \\ 0 & 0.364 & 1.64 \end{bmatrix}\begin{Bmatrix} A_x \\ A_\theta \\ A_\phi \end{Bmatrix}$$

After several iterations, we will arrive at the result

$$\{N\} = 0.167\frac{m}{k}\begin{Bmatrix} 0.00 \\ 1.00 \\ 1 \end{Bmatrix}$$

Then the second eigenvector and eigenvalue of the system are

$$\{\phi\} = \begin{Bmatrix} 0.00 \\ 1.00 \\ 1 \end{Bmatrix}$$

$$\omega_2^2 = \frac{k}{0.167m} = 6.00\frac{k}{m}$$

which compare with the results of Example 5 4

(c) *An Alternate Approach for the Higher Modes, Systems with Rigid-Body Degrees of Freedom*

Let us consider an alternate method for obtaining the higher eigenvectors and eigenvalues which does not involve the insertion of sweeping matrices.

Partitioning the matrices of Eq. (5.27) and arbitrarily dropping the first row of the coefficient matrices, we can write

$$-\omega^2\left[\begin{array}{c:cccc} m_{21} & m_{22} & m_{23} & \cdots & m_{2n} \\ m_{31} & m_{32} & m_{33} & \cdots & \cdot \\ \vdots & \vdots & & & \vdots \\ m_{n1} & m_{n2} & \cdot & \cdots & m_{nn} \end{array}\right]\left\{\begin{array}{c} A_1 \\ \hdashline A_2 \\ A_3 \\ \vdots \\ A_n \end{array}\right\} + \left[\begin{array}{c:cccc} k_{21} & k_{22} & k_{23} & \cdots & k_{2n} \\ k_{31} & k_{32} & k_{33} & \cdots & \cdot \\ \vdots & \vdots & & & \vdots \\ k_{n1} & k_{n2} & \cdot & \cdots & k_{nn} \end{array}\right]\left\{\begin{array}{c} A_1 \\ \hdashline A_2 \\ A_3 \\ \vdots \\ A_n \end{array}\right\} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \tag{6.30}$$

If the first eigenvector has been found, we can relate one of the amplitudes, in this case A_1, to the other amplitudes as in Eq. (6.23), written as

$$A_1 = \lfloor S_{12}\, S_{13} \cdots S_{1n} \rfloor \begin{Bmatrix} A_2 \\ A_3 \\ \vdots \\ A_n \end{Bmatrix} \tag{6.31}$$

Substitution of Eq. (6.31) into Eq. (6.30) leads to

$$-\omega^2\left[\begin{Bmatrix} m_{21} \\ m_{31} \\ \vdots \\ m_{n1} \end{Bmatrix}\lfloor S_{12} \cdots S_{1n} \rfloor + \begin{bmatrix} m_{22} & m_{23} & \cdots & m_{2n} \\ m_{32} & m_{33} & \cdots & \cdot \\ \vdots & & & \vdots \\ m_{n2} & \cdot & \cdots & m_{nn} \end{bmatrix}\right]\begin{Bmatrix} A_2 \\ A_3 \\ \vdots \\ A_n \end{Bmatrix} + \left[\begin{Bmatrix} k_{21} \\ k_{31} \\ \vdots \\ k_{n1} \end{Bmatrix}\lfloor S_{12} \cdots S_{1n} \rfloor + \begin{bmatrix} k_{22} & k_{23} & \cdots & k_{2n} \\ k_{32} & k_{33} & \cdots & \cdot \\ \vdots & & & \vdots \\ k_{n2} & \cdot & \cdots & k_{nn} \end{bmatrix}\right]\begin{Bmatrix} A_2 \\ A_3 \\ \vdots \\ A_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \tag{6.32}$$

Assuming the modified stiffness matrix of Eq. (6.32) to be nonsingular, we can premultiply the equations by its inverse, leading to

$$\frac{1}{\omega^2}\begin{Bmatrix} A_2 \\ A_3 \\ \vdots \\ A_n \end{Bmatrix} + \left[\begin{Bmatrix} k_{21} \\ k_{31} \\ \vdots \\ k_{n1} \end{Bmatrix}\lfloor S_{12} \cdots S_{1n} \rfloor + \begin{bmatrix} k_{22} & k_{23} & \cdots & k_{2n} \\ k_{32} & k_{33} & \cdots & \cdot \\ \vdots & & & \vdots \\ k_{n2} & \cdot & \cdots & k_{nn} \end{bmatrix}\right]^{-1} \cdot \left[\begin{Bmatrix} m_{21} \\ m_{31} \\ \vdots \\ m_{n1} \end{Bmatrix}\lfloor S_{12} \cdots S_{1n} \rfloor + \begin{bmatrix} m_{22} & m_{23} & \cdots & m_{2n} \\ m_{32} & m_{33} & \cdots & \cdot \\ \vdots & & & \vdots \\ m_{n2} & \cdot & & m_{nn} \end{bmatrix}\right]\begin{Bmatrix} A_2 \\ A_3 \\ \vdots \\ A_n \end{Bmatrix} \tag{6.33}$$

Equation (6.33) represents a set of $n-1$ equations in the $n-1$ remaining eigenvectors and associated eigenvalues. The second eigenvector and eigenvalue of the system may be obtained by an iterative procedure based on Eq. (6.33). If the modified stiffness matrix of Eq. (6.32) is singular, we can always form a nonsingular matrix of the same form from another set of $n-1$ rows and columns

Knowing two of the eigenvectors, two of the amplitudes, arbitrarily A_1 and A_2, can be related to the remaining amplitudes as in Eq. (6 27). Making use of Eq. (6.27), and dropping the first two rows of the coefficient matrices of Eq. (5.27), we can write

$$-\omega^2\left[\begin{bmatrix} m_{31} & m_{32} \\ m_{41} & m_{42} \\ \cdot & \\ m_{n1} & m_{n2}\end{bmatrix}\begin{bmatrix} S_{13} & \cdots & S_{1n} \\ S_{23} & \cdot & S_{2n}\end{bmatrix}+\begin{bmatrix} m_{33} & m_{34} & \cdots & m_{3n} \\ m_{43} & m_{44} & & \\ & & & \\ m_{n3} & \cdot & \cdots & m_{nn}\end{bmatrix}\right]\begin{Bmatrix} A_3 \\ A_4 \\ \vdots \\ A_n\end{Bmatrix}$$
$$+\left[\begin{bmatrix} k_{31} & k_{32} \\ k_{41} & k_{42} \\ \cdot & \cdot \\ k_{n1} & k_{n2}\end{bmatrix}\begin{bmatrix} S_{13} & \cdot & S_{1n} \\ S_{23} & \cdot & S_{2n}\end{bmatrix}\right.$$
$$\left.+\begin{bmatrix} k_{33} & k_{34} & \cdots & k_{3n} \\ k_{43} & k_{44} & \cdots & \cdot \\ & & & \vdots \\ k_{n3} & & \cdots & k_{nn}\end{bmatrix}\right]\begin{Bmatrix} A_3 \\ A_4 \\ \vdots \\ A_n\end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ \\ 0\end{Bmatrix} \tag{6.34}$$

If the modified stiffness matrix of Eq. (6.34) is nonsingular, we can write

$$\frac{1}{\omega^2}\begin{Bmatrix} A_3 \\ A_4 \\ \\ A_n\end{Bmatrix}=\left[\begin{bmatrix} k_{31} & k_{32} \\ k_{41} & k_{42} \\ \cdot & \\ k_{n1} & k_{n2}\end{bmatrix}\begin{bmatrix} S_{13} & & S_{1n} \\ S_{23} & \cdot & S_{2n}\end{bmatrix}+\begin{bmatrix} k_{33} & k_{34} & & k_{3n} \\ k_{43} & k_{44} & \cdots & \\ \cdot & & & \cdot \\ k_{n3} & & & k_{nn}\end{bmatrix}\right]^{-1}$$
$$\cdot\left[\begin{bmatrix} m_{31} & m_{32} \\ m_{41} & m_{42} \\ \cdot & \\ m_{n1} & m_{n2}\end{bmatrix}\begin{bmatrix} S_{13} & \cdots & S_{1n} \\ S_{23} & & S_{2n}\end{bmatrix}+\begin{bmatrix} m_{33} & m_{34} & \cdots & m_{3n} \\ m_{43} & m_{44} & & \\ & & & \cdot \\ m_{n3} & & & m_{nn}\end{bmatrix}\right]\begin{Bmatrix} A_3 \\ A_4 \\ \\ A_n\end{Bmatrix} \tag{6.35}$$

This is a set of $n-2$ equations in the remaining $n-2$ eigenvectors and eigenvalues The third eigenvector and eigenvalue of the system can be obtained by an iterative procedure based on Eq. (6 35)

The procedure just outlined is particularly useful for systems which have rigid-body degrees of freedom For this case the stiffness matrix is singular and Eq. (6.1) cannot be written. However the mode shapes or eigenvectors associated with the rigid-body degrees of freedom can generally be written

from inspection of the system. Then equations such as Eqs. (6.33) and (6.35) can be written.

EXAMPLE 6.3

The system illustrated by Fig. 5-4 has one rigid-body degree of freedom. By inspection, the eigenvector describing rigid-body motion is given by

$$\{\phi\}_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

The two remaining eigenvectors must be orthogonal with the rigid-body eigenvector with respect to the inertia matrix as expressed by

$$\lfloor 1\ 1\ 1 \rfloor \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = 0$$

or by

$$A_1 = \lfloor -1\ -1 \rfloor \begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix}$$

From the results of Example 5.5, we can write

$$-\omega^2 \left[\begin{array}{c|cc} m & 0 & 0 \\ \hline 0 & m & 0 \\ 0 & 0 & m \end{array}\right] \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} + \left[\begin{array}{c|cc} k & -k & 0 \\ \hline -k & 2k & -k \\ 0 & -k & k \end{array}\right] \begin{Bmatrix} A_1 \\ \hline A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Dropping the first row of the coefficient matrices and making use of the expression for A_1, we can write

$$-\omega^2 \left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \lfloor -1\ -1 \rfloor + \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \right] \begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix}$$

$$+ \left[\begin{Bmatrix} -k \\ 0 \end{Bmatrix} \lfloor -1\ -1 \rfloor + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \right] \begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

or

$$-\omega^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix} + \begin{bmatrix} 3k & 0 \\ -k & k \end{bmatrix} \begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Premultiplying by the inverse of the modified stiffness matrix, we can write

$$\frac{1}{\omega^2} \begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix} = \frac{m}{3k} \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix}$$

The second eigenvector and eigenvalue can be obtained using an iterative procedure based on this equation.

6.2 The Tabular Method of Holzer

(a) Torsional Vibrations

The method described below is especially useful for systems in which the mass elements are arranged in a row. Such systems are quite common. For example, the mass elements of a beam or rod are arranged in a row. The method considered here was developed by Holzer (Ref. 11) to determine the torsional natural frequencies of crankshafts.

Let us consider the free torsional vibration of a set of n disks on an elastic shaft as shown in Fig. 6-1. The n coordinates of angular motion $\theta_1, \theta_2, \ldots, \theta_n$,

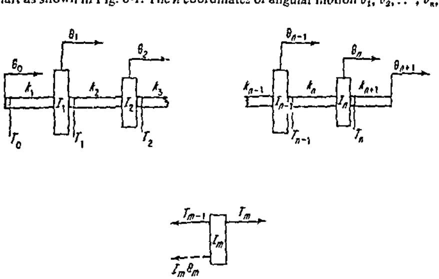


Fig. 6-1 *Torsional system of shafts and disks*

describe the torsional motion of the system. The axial moments of inertia of the disks are represented by $I_1, I_2, \ldots, I_n$ and the torsional stiffnesses by $k_1, k_2, \ldots, k_{n+1}$. The torques in the shaft are given by $T_0, T_1, \ldots, T_n$. For built-in ends of the shaft, we will require $\theta_0 = \theta_{n+1} = 0$. Similarly, for free ends, we will specify $T_0 = T_n = 0$. Referring to the free-body diagram of Fig. 6-1, we can write the requirements for dynamic equilibrium of the system as

$$\begin{aligned}
-I_1\ddot{\theta}_1 + T_1 - T_0 &= 0 \\
-I_2\ddot{\theta}_2 + T_2 - T_1 &= 0 \\
&\vdots \\
-I_n\ddot{\theta}_n + T_n - T_{n-1} &= 0
\end{aligned} \tag{6.36}$$

Further, the relationship between the torques and the relative rotations of the disks are given by

$$\begin{aligned} T_0 - k_1\ (\theta_1 - \theta_0) &= 0 \\ T_1 - k_2\ (\theta_2 - \theta_1) &= 0 \\ \vdots \quad \vdots \quad \vdots \quad &\vdots \\ T_n - k_{n+1}(\theta_{n+1} - \theta_n) &= 0 \end{aligned} \tag{6.37}$$

If the system is in a simple harmonic motion of frequency ω, we can write

$$\begin{aligned} \ddot{\theta}_1 &= -\omega^2\theta_1 \\ \ddot{\theta}_2 &= -\omega^2\theta_2 \\ \vdots \quad &\quad \vdots \\ \ddot{\theta}_n &= -\omega^2\theta_n \end{aligned} \tag{6.38}$$

The three sets of equations just given serve as the basis for Holzer's method.

The method developed by Holzer involves an iterative procedure we can describe as a trial eigenvalue method. Any of the eigenvalues ω^2 of the system will satisfy Eqs. (6.36) through (6.38) as well as the appropriate boundary conditions. Suppose that ω^2 represents a trial eigenvalue. Let us require the equations of motion and the boundary condition at the left-hand end to be satisfied by our choice for ω^2. Then, unless ω^2 happens to be an exact eigenvalue of the system, we will find that the boundary condition at the right-hand end is not satisfied. If the left-hand end is built-in, we will require $\theta_0 = 0$ and will select an arbitrary value for T_0. If the left-hand end is free, we will require $T_0 = 0$ and will select an arbitrary value for θ_0. For either case, we can write

$$\begin{aligned} \theta_1 &= \theta_0 + \frac{T_0}{k_1} \\ T_1 &= T_0 - I_1\omega^2\theta_1 \end{aligned} \tag{6.39}$$

in which we have used the first equation in each of Eqs. (6.36) through (6.38). Having determined the conditions at the first disk, we can proceed to determine those at the second disk, leading to

$$\begin{aligned} \theta_2 &= \theta_1 + \frac{T_1}{k_2} \\ T_2 &= T_1 - I_2\omega^2\theta_2 = T_0 - \omega^2 \sum_{m=1}^{2} I_m\theta_m \end{aligned} \tag{6.40}$$

The second equation of each of Eqs. (6.36) through (6.38) has been used. Continuing the step-by-step procedure, we will arrive at the conditions at the nth disk, given by

$$\begin{aligned} \theta_n &= \theta_{n-1} + \frac{T_{n-1}}{k_n} \\ T_n &= T_{n-1} - I_n\omega^2\theta_n = T_0 - \omega^2 \sum_{m=1}^{n} I_m\theta_m \end{aligned} \tag{6.41}$$

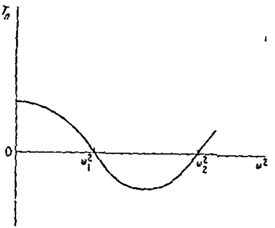


Fig. 6-2 Residual torque at a free end.

If the right-hand end is a free end, the torque T_n should be zero. If it is not, our trial eigenvalue is not an eigenvalue of the system. Then we can choose another value for our trial eigenvalue and repeat the procedure. It is convenient to plot the residual torque T_n as a function of the trial eigenvalue as in Fig. 6-2. Evidently ω_1^2 and ω_2^2 represent the first two eigenvalues of the system. If the right-hand end is built-in, we can write

$$\theta_{n+1} = \theta_n + \frac{T_n}{k_{n+1}} \tag{6.42}$$

If the rotation θ_{n+1} is not zero, our trial eigenvalue is not one of the system eigenvalues. Further trials will lead us to improved estimates for the eigenvalues. The plot for the residual rotation θ_{n+1} as a function of the trial eigenvalue looks much like Fig. 6-2.

Consider the free longitudinal vibration of a set of n masses restrained by elastic springs as shown in Fig. 6-3. Comparison of Fig. 6-1 and 6-3 reveals

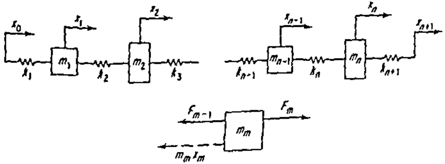


Fig. 6-3 Longitudinal system of masses and springs.

that the torsional and longitudinal systems are similar. We can apply Holzer's method as outlined to the longitudinal system if we replace the coordinates θ by x, the moments of inertia I by the masses m, and the torques T by the forces F.

EXAMPLE 6.4

A uniform circular bar, built in at one end and free at the other, is idealized as a lumped mass bar with four lumps as shown in Fig. 6-4. Assuming that

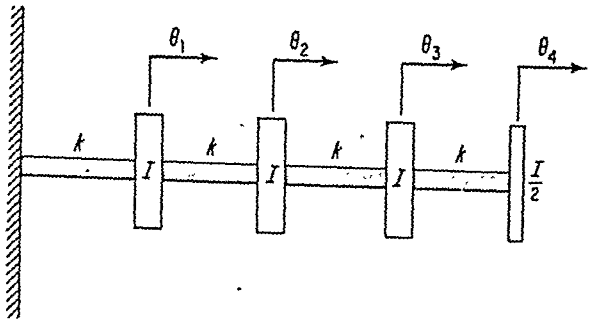


Fig. 6-4

we are interested in torsional vibrations, the axial moment of inertia of each of the four equal sections has been lumped equally into the disks at the ends of each section. Thus I represents one-fourth of the total axial moment of inertia of the bar. The torsional stiffness of each of the four sections is k.

Let us determine the fundamental natural frequency, using Holzer's method. Since the left-hand end of the bar is built-in, we can write that $\theta_0 = 0$ and arbitrarily that $T_0/k = 1$. As a first trial, let us use $I\omega^2/k = 0.15$. The calculations are shown in tabular form in Table 6-1A. The residual torque at the free end is given by $T_4/k = 0.0177$.

Table 6-1A
HOLZER METHOD CALCULATIONS

① m	② $\dfrac{I_m\omega^2}{k}$	③ θ_m ③$_{m-1}$ + ⑦$_{m-1}$	④ $\dfrac{\Delta T}{k}$ − ② × ③	⑤ $\dfrac{T_m}{k}$ ⑤$_{m-1}$ + ④$_m$	⑥ $\dfrac{k_{m+1}}{k}$	⑦ $\Delta\theta$ ⑤/⑥
0		0		1		
1	0.15	1	−0.15	0.85	1	1
2	0.15	1.85	−0.2775	0.5725	1	0.85
3	0.15	2.4225	−0.3634	0.2091	1	0.5725
4	0.075	2.6316	−0.1974	0.0177	1	0.2091

Trial $\dfrac{I\omega^2}{k} = 0.15$

Table 6-1B

HOLZER METHOD CALCULATIONS

①	②	③	④	⑤	⑥	⑦
m	$\frac{I_m\omega^2}{k}$	θ_m	$\frac{\Delta T}{k}$	$\frac{T_m}{k}$	$\frac{k_{m+1}}{k}$	$\Delta\theta$
		③$_{m-1}$ + ⑦$_{m-1}$	− ② × ③	⑤$_{m-1}$ + ④$_m$		⑤/⑥
0		0		1	1	1
1	0.155	1	−0.155	0.845	1	0.845
2	0.155	1.845	−0.2860	0.5590	1	0.5590
3	0.155	2.4040	−0.3726	0.1864	1	0.1864
4	0.0775	2.5904	−0.2008	−0.0144		

$$\text{Trial } \frac{I\omega^2}{k} = 0.155$$

Next let us try $I\omega^2/k = 0.155$. The calculations for the second trial are given in Table 6-1B. For the second trial, the residual torque is given by $T_4/k = -0.0144$. Evidently the eigenvalue lies between the two trials. A straight-line interpolation yields the estimate for the eigenvalue $I\omega^2/k = 0.1522$. The residual torque at the free end T_4/k is shown in Fig. 6-5 as a function of the trial eigenvalue ω^2. Note that the fundamental mode shape can be obtained from column ③ by interpolating between parts A and B of Table 6-1. Similarly the torque distribution in the fundamental mode may be obtained from column ⑤.

In this example the trial eigenvalues were selected near the exact value of the eigenvalue. This made it possible to bracket the eigenvalue closely with only two trials. Generally it will be necessary to make a number of trials.

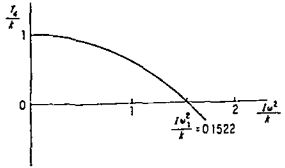


Fig. 6-5 *Residual torque at the free end of the system of Fig. 6-4*

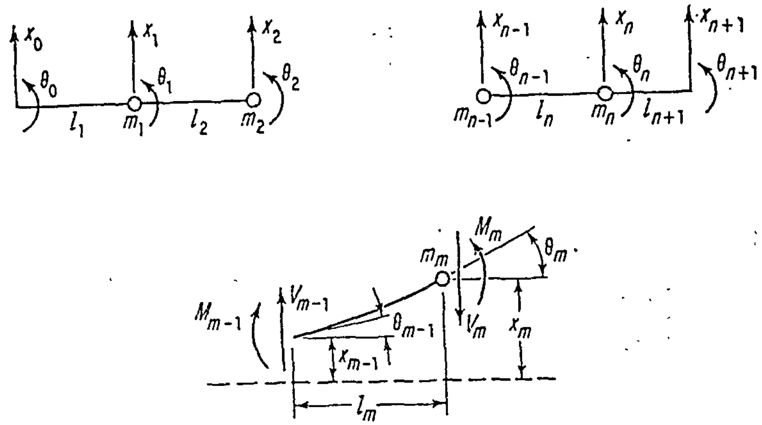


Fig. 6-6 *Lumped-mass beam.*

(*b*) *Flexural Vibrations of a Beam*

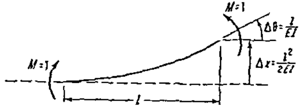
The method outlined above for torsional and longitudinal vibrations has been adapted by Myklestad (Ref. 12) and Prohl (Ref. 13) to the analysis of the flexural vibrations of a lumped mass beam. Consider the idealized beam of Fig. 6-6 in which the mass of the beam is concentrated into n particles connected by massless segments having the elastic properties of the beam. The forces and moments acting on the mth segment of the beam including the mth mass are shown in the free-body diagram. For simple harmonic motion of frequency ω, we can replace $\ddot{x}_m$ by $-\omega^2 x_m$. Dynamic equilibrium of the mth segment is assured if

$$\begin{aligned} M_m &= M_{m-1} + V_{m-1} l_m \\ V_m &= V_{m-1} + m_m \omega^2 x_m \end{aligned} \tag{6.43}$$

For a uniform beam segment of length l and flexural rigidity EI, Fig. 6-7 shows the relative displacement and rotation of the ends resulting from a unit moment or unit force applied at the left-hand end. Making use of this information, we can write expressions for the displacement and rotation of the right-hand end of the beam segment as

$$\begin{aligned} x_m &= x_{m-1} + l_m\theta_{m-1} + M_{m-1}\frac{l_m^2}{2(EI)_m} + V_{m-1}\frac{l_m^3}{6(EI)_m} \\ \theta_m &= \theta_{m-1} + M_{m-1}\frac{l_m}{(EI)_m} + V_{m-1}\frac{l_m^2}{2(EI)_m} \end{aligned} \tag{6.44}$$

Starting from the left-hand end we can determine the quantities x, θ, V, M successively from station to station along the beam, using Eqs. (6.43) and (6.44). From the boundary conditions two of the quantities x, θ, m, V will be



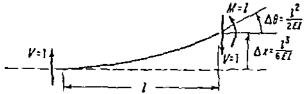


Fig. 6-7 *Deformation of a beam segment*

known at the left-hand end. Arbitrary values can be selected for the remaining two quantities. The two boundary conditions at the right-hand end will not be satisfied unless the trial eigenvalue ω^2 is equal to one of the eigenvalues of the beam Further, it is necessary that the arbitrary values for the two quantities at the left-hand end have a definite ratio

The significant advantage of Holzer's method lies in the fact that the higher eigenvalues can be obtained as easily and with as much accuracy as the fundamental eigenvalue Ordinarily the method is used for systems in which the mass elements are arranged in a row However, the method can be adapted to the analysis of branched systems For further elaboration on Holzer's method, consult Refs 14 and 15

6 3 Theory of the Transfer Matrix

(a) *Torsional Vibrations*

For the torsional system of Fig 6-1, the *m*th set of equations having the form of Eqs. (6 39) through (6.41) can be written in the matrix form

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_m = \begin{bmatrix} 1 & \dfrac{1}{k} \\ -I\omega^2 & 1 - \dfrac{I\omega^2}{k} \end{bmatrix}_m \begin{Bmatrix} \theta \\ T \end{Bmatrix}_{m-1} \tag{6.45}$$

The column matrix $\begin{Bmatrix} \theta \\ T \end{Bmatrix}$ is referred to as a state vector while the square matrix is a transfer matrix. Thus premultiplication of the $(m-1)$th state vector

by the mth transfer matrix results in the mth state vector. With a knowledge of the transfer matrices of the system, we can relate the state vectors at the left- and right-hand ends of the system by

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_n = \begin{bmatrix} 1 & \dfrac{1}{k} \\ -I\omega^2 & 1 - \dfrac{I\omega^2}{k} \end{bmatrix}_n \cdots \begin{bmatrix} 1 & \dfrac{1}{k} \\ -I\omega^2 & 1 - \dfrac{I\omega^2}{k} \end{bmatrix}_1 \begin{Bmatrix} \theta \\ T \end{Bmatrix}_0 \tag{6.46}$$

For the usual boundary conditions at the left-hand end of the shaft, we can arbitrarily write either

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_0 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

or (6.47)

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

representing a free end or a built-in end. At the right-hand end, either the rotation θ_{n+1} or the torque T_n will be required to be zero. We can use the matrix formulation just given in performing the calculations needed by the iterative method of Holzer. As an alternate method, we can obtain the characteristic equation of the system from the satisfaction of the boundary conditions at the right-hand end. The latter method is illustrated in the following example.

EXAMPLE 6.5

For the system of Fig. 6-4, the equations having the form of Eq. (6.45) are given by

$$\begin{Bmatrix} \theta_1 \\ \dfrac{T_1}{k} \end{Bmatrix} = \begin{bmatrix} 1 & \dfrac{1}{k} \\ -\dfrac{I\omega^2}{k} & 1 - \dfrac{I\omega^2}{k} \end{bmatrix} \begin{Bmatrix} \theta_0 \\ \dfrac{T_0}{k} \end{Bmatrix}$$

$$\begin{Bmatrix} \theta_2 \\ \dfrac{T_2}{k} \end{Bmatrix} = \begin{bmatrix} 1 & \dfrac{1}{k} \\ -\dfrac{I\omega^2}{k} & 1 - \dfrac{I\omega^2}{k} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \dfrac{T_1}{k} \end{Bmatrix}$$

$$\begin{Bmatrix} \theta_3 \\ \dfrac{T_3}{k} \end{Bmatrix} = \begin{bmatrix} 1 & \dfrac{1}{k} \\ -\dfrac{I\omega^2}{k} & 1 - \dfrac{I\omega^2}{k} \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \dfrac{T_2}{k} \end{Bmatrix}$$

$$\begin{Bmatrix} \theta_4 \\ \dfrac{T_4}{k} \end{Bmatrix} = \begin{bmatrix} 1 & \dfrac{1}{k} \\ -\dfrac{I\omega^2}{2k} & 1 - \dfrac{I\omega^2}{2k} \end{bmatrix} \begin{Bmatrix} \theta_3 \\ \dfrac{T_3}{k} \end{Bmatrix}$$

As in Example 6.4, we will write the boundary condition at the left-hand end as

$$\begin{Bmatrix} \theta_0 \\ \dfrac{T_0}{k} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

At the right-hand end, the torque T_4/k must be zero. Combination of the equations leads to

$$\frac{T_4}{k} = 0 = \left\lfloor -\frac{I\omega^2}{2k}, 1 - \frac{I\omega^2}{2k} \right\rfloor \begin{bmatrix} 1 & \dfrac{1}{k} \\ -\dfrac{I\omega^2}{k} & 1 - \dfrac{I\omega^2}{k} \end{bmatrix}^3 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

in which the square matrix is cubed Expansion of this equation leads to the characteristic equation from which the four natural frequencies may be obtained.

(*b*) *Flexural Vibrations of a Beam*

For the idealized beam of Fig. 6-6, we can write Eqs. (6 43) and (6.44) in the matrix form

$$\begin{Bmatrix} x \\ \theta \\ M \\ V \end{Bmatrix}_m = \begin{bmatrix} 1 & l & \dfrac{l^2}{2EI} & \dfrac{l^3}{6EI} \\ 0 & 1 & \dfrac{l}{EI} & \dfrac{l^2}{2EI} \\ 0 & 0 & 1 & l \\ m\omega^2 & ml\omega^2 & \dfrac{ml^2\omega^2}{2EI} & 1 + \dfrac{ml^3\omega^2}{6EI} \end{bmatrix}_m \begin{Bmatrix} x \\ \theta \\ M \\ V \end{Bmatrix}_{m-1} \tag{6.48}$$

In arriving at these equations, the expression for x_m has been substituted into the right-hand side of the expression for V_m The column matrices are the mth and $(m-1)$th state vectors and the square matrix is the mth transfer matrix. We can relate the state vectors at the two ends of the beam by

$$\begin{Bmatrix} x \\ \theta \\ M \\ V \end{Bmatrix}_n = [T]_n[T]_{n-1} \cdots [T]_1 \begin{Bmatrix} x \\ \theta \\ M \\ V \end{Bmatrix}_0 \tag{6.49}$$

Two boundary conditions must be satisfied at each end of the beam. As shown in the following example we can obtain the characteristic equation of the system from Eq. (6 49) by requiring satisfaction of the boundary conditions

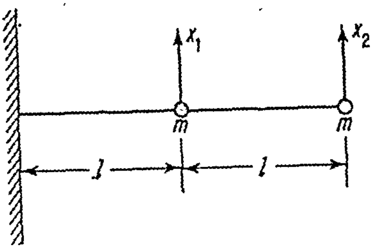


Fig. 6-8

EXAMPLE 6.6

A cantilever beam is idealized as a uniform elastic beam supporting two particles of mass m as shown in Fig. 6-8. At the root, the boundary conditions are given by $x_0 = \theta_0 = 0$. Making use of Eqs. (6.48) and (6.49), we can write expressions for the moment and shear force at the free end, as given by

$$\begin{Bmatrix} M_2 \\ V_2 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & l \\ m\omega^2 & ml\omega^2 & \dfrac{ml^2\omega^2}{2EI} & 1 + \dfrac{ml^3\omega^2}{6EI} \end{bmatrix} \begin{bmatrix} \dfrac{l^2}{2EI} & \dfrac{l^3}{6EI} \\ \dfrac{l}{EI} & \dfrac{l^2}{2EI} \\ 1 & l \\ \dfrac{ml^2\omega^2}{2EI} & 1 + \dfrac{ml^3\omega^2}{6EI} \end{bmatrix} \begin{Bmatrix} M_0 \\ V_0 \end{Bmatrix}$$

Simplifying, we can write

$$\begin{Bmatrix} M_2 \\ V_2 \end{Bmatrix} = \begin{bmatrix} 1 + \dfrac{ml^3\omega^2}{2EI} & 2l + \dfrac{ml^4\omega^2}{6EI} \\ \dfrac{5ml^2\omega^2}{2EI} + \dfrac{m^2l^5\omega^4}{12(EI)^2} & \dfrac{7ml^3\omega^2}{6EI} + \left(1 + \dfrac{ml^3\omega^2}{6EI}\right)^2 \end{bmatrix} \begin{Bmatrix} M_0 \\ V_0 \end{Bmatrix}.$$

Since the moment and shear force at a free end must be zero while those at the root are not, the determinant of the square matrix must be zero. Expanding the determinant, we can write the characteristic equation

$$\omega^4 - \frac{108}{7}\cdot\frac{EI}{ml^3}\,\omega^2 + \frac{36}{7}\cdot\frac{(EI)^2}{m^2l^6} = 0$$

The eigenvalues of the system are

$$\omega_1^2 = 0.341\,\frac{EI}{ml^3}$$

$$\omega_2^2 = 15.1\,\frac{EI}{ml^3}$$

Substitution of the eigenvalues into either of the equations in M_0 and V_0 yields the ratios

$$\left(\frac{M_0}{V_0}\right)^{(1)} = -1.76l$$

$$\left(\frac{M_0}{V_0}\right)^{(2)} = -0.529l$$

for the first and second modes. If we normalize to unit shear force at the root the state vectors at the root for the first and second modes are given by

$$\begin{Bmatrix} x_0 \\ \theta_0 \\ M_0 \\ V_0 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0 \\ 0 \\ -1.76l \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} x_0 \\ \theta_0 \\ M_0 \\ V_0 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0 \\ 0 \\ -0.529l \\ 1 \end{Bmatrix}$$

Starting with these results, we can obtain the state vectors at the other two points for the first and second modes.

6.4 Approximate Methods Based on the Energy Approach

If each of the n coordinates $x_1, x_2, \ldots, x_n$ represents one of the degrees of freedom of one of the lumped masses, we will generally need all n of the coordinates to adequately describe the motion. On the other hand, if the coordinates are generalized coordinates which represent simultaneous motion of all of the lumped masses of the system, it will often be the case that we can closely approximate the motion with fewer than n coordinates This is particularly true for the choice of normal coordinates

With experience in analyzing real problems, we will find that the representation of the motion in terms of the normal modes is usually quite convergent with the mode number Often much of the motion of the system occurs in the fundamental mode or in the first few modes. The basic reason for this behavior is that the stiffness of the modes increases with the mode number. If the mode shapes are selected such that the maximum amplitudes in all of the modes are of the same order of magnitude, we can expect from Eq. (5.52) that the generalized masses will be of the same order of magnitude Since the natural frequencies increase with the mode number, it is evident from Eq (5.57) that the generalized stiffnesses will increase with the mode number. We

For a free vibration of the system, we can try the solution

$$\{p\} = \{A\} \sin \omega t \tag{6.65}$$

Substitution of the trial solution into the equations of motion leads to

$$[[K] - \omega^2[M]]\{A\} = \{0\} \tag{6.66}$$

and to the characteristic equation

$$|[K] - \omega^2[M]| = 0 \tag{6.67}$$

Solution of this equation yields the k eigenvalues $\omega_1^2, \omega_2^2, \ldots, \omega_k^2$. The relative amplitudes corresponding to each of the eigenvalues are obtained from Eq. (6.66). Then we can write the general solution for a free vibration of the system as

$$\begin{Bmatrix} p_1 \\ p_2 \\ \vdots \\ p_k \end{Bmatrix} = \begin{bmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{1k} \\ \psi_{21} & \psi_{22} & \cdots & \cdot \\ \vdots & & & \vdots \\ \psi_{k1} & \cdot & \cdots & \psi_{kk} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_k \end{Bmatrix} \tag{6.68}$$

in which

$$\begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_k \end{Bmatrix} = \begin{Bmatrix} C_1 \sin \omega_1 t + C_1' \cos \omega_1 t \\ C_2 \sin \omega_2 t + C_2' \cos \omega_2 t \\ \vdots \qquad\qquad \vdots \\ C_k \sin \omega_k t + C_k' \cos \omega_k t \end{Bmatrix} \tag{6.69}$$

The columns $\{\psi\}$ represent the normal mode shapes corresponding to the approximate representation of the motion given by Eq. (6.58). Making use of Eqs. (6.58) and (6.68), we can write the equations of transformation relating the original coordinates $\{x\}$ and the reduced number of normal coordinates $\{q\}$ by

$$\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1k} \\ \phi_{21} & \phi_{22} & \cdots & \cdot \\ \vdots & & & \vdots \\ \phi_{n1} & \cdot & \cdots & \phi_{nk} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_k \end{Bmatrix} \tag{6.70}$$

in which

$$[\phi] = [\gamma][\psi] \tag{6.71}$$

It is not difficult to show that the normal mode shapes $\{\psi\}$ are orthogonal with respect to the inertia and stiffness matrices $[M]$ and $[K]$ of Eq. (6.60), as given by

$$\begin{aligned} \lfloor\psi\rfloor_i[M]\{\psi\}_j &= 0 \\ \lfloor\psi\rfloor_i[K]\{\psi\}_j &= 0 \end{aligned} \tag{6.72}$$

provided that $i \neq j$ and that the eigenvalues are distinct. Making use of Eqs (6.60) and (6.71), we can write

$$\begin{aligned} \lfloor\phi\rfloor_i[m]\{\phi\}_j &= 0 \\ \lfloor\phi\rfloor_i[k]\{\phi\}_j &= 0 \end{aligned} \tag{6.73}$$

if $i \neq j$. This is a statement that the normal mode shapes $\{\phi\}$ are orthogonal with respect to the original inertia and stiffness matrices $[m]$ and $[k]$.

Let us write the equations of motion in terms of the normal coordinates. Replacing the equations of transformation of Eq. (6.58) by those of Eq. (6.70), we can write the expressions for the generalized masses and stiffnesses, given earlier by Eqs. (6.60), as

$$\begin{aligned} [M] &= [\phi]^T[m][\phi] \\ [K] &= [\phi]^T[k][\phi] \end{aligned} \tag{6.74}$$

The inertia and stiffness matrices are diagonal matrices as indicated as a result of the orthogonality relations, Eqs. (6.73). Then the generalized forces, given earlier by Eqs. (6.61) and (6.63), are given by

$$\begin{aligned} \{Q\}_{in} &= -[M]\{\ddot{q}\} \\ \{Q\}_{el} &= -[K]\{q\} \\ \{Q\}_{ex} &= [\phi]^T\{F\}_{ex} \end{aligned} \tag{6.75}$$

The equations of motion are

$$\{\Sigma\, Q\} = -[M]\{\ddot{q}\} - [K]\{q\} + \{Q\}_{ex} = \{0\} \tag{6.76}$$

Since the inertia and stiffness matrices are diagonal matrices, the equations of motion are uncoupled.

The success of the procedure just outlined in approximating the motion of a system depends on the care with which the shapes $\{y\}$ have been selected.

EXAMPLE 6.8

For the system of Fig. 6-4, let us approximate the motion with the two coordinates defined by the equations of transformation

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{16} \\ \frac{1}{2} & \frac{1}{4} \\ \frac{3}{4} & \frac{9}{16} \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$$

The shapes associated with the two coordinates were obtained from the two terms of a power series x and x^2, x being the length along the bar with origin at the built-in end. Neither of the shapes approximates the fundamental

Note that this result approximates the exact fundamental mode shape more closely than the single shape assumed in Example 6.7. Further, the estimate of the fundamental frequency is closer.

6.5 Approximate Methods Based on the Equations of Motion

In the preceding section, we considered the approximate representation of the motion of a system with a reduced number of coordinates. The reduced number of equations of motion were obtained using an energy method. Using the same type of approximate representation, let us write the reduced set of equations starting with the original set of equations of motion.

(a) *Reduction to a Single Generalized Coordinate*

The relation between the original coordinates $\{x\}$ and a single generalized coordinate p was given by Eq. (6.50). Substitution of Eq. (6.50) into the equations of motion for the system, Eq. (5.24), leads to

$$\begin{Bmatrix} \sum F_1 \\ \sum F_2 \\ \vdots \\ \sum F_n \end{Bmatrix} = -\begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & \cdot \\ \vdots & & & \vdots \\ m_{n1} & \cdot & \cdots & m_{nn} \end{bmatrix} \begin{Bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{Bmatrix} \ddot{p}$$
$$- \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & \cdot \\ \vdots & & & \vdots \\ k_{n1} & & \cdots & k_{nn} \end{bmatrix} \begin{Bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{Bmatrix} p + \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix}_{ex} = \begin{Bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{Bmatrix} \tag{6.77}$$

The ith row of this equation represents the sum of the generalized forces $\sum F_i$ associated with the coordinate x_i, but which result from motion of the coordinate p. Since the representation of the motion by a single coordinate is approximate, the forces will not in general be in equilibrium. The quantity e_i on the right-hand side of Eq. (6.77) represents the error in equilibrium of the forces associated with x_i.

The coordinate p cannot be required to satisfy more than one differential equation. We can obtain a single equation of motion by requiring that the forces associated with x_i be in equilibrium, as given by $e_i = 0$. Then we can write

$$\sum F_i = -\lfloor m_{i1} \cdots m_{in} \rfloor \begin{Bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{Bmatrix} \ddot{p} - \lfloor k_{i1} \cdots k_{in} \rfloor \begin{Bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{Bmatrix} p + F_{i,ex} = 0 \tag{6.78}$$

When applied to a continuous system, the procedure just described in which the solution is required to satisfy the conditions of equilibrium at a reduced

number of points of the system is referred to as collocation. For a free vibration of the system, we can obtain from Eq. (6.78) an approximation for the fundamental eigenvalue, given by

$$\omega^2 = \frac{\lfloor k_{i1} \cdots k_{in} \rfloor \begin{Bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{Bmatrix}}{\lfloor m_{i1} \cdots m_{in} \rfloor \begin{Bmatrix} \gamma_1 \\ \cdot \\ \gamma_n \end{Bmatrix}} \tag{6.79}$$

Corresponding to $i = 1, 2, \ldots, n$ the method of collocation can lead to n different results of the form of Eqs. (6.78) and (6.79). If $\{\gamma\}$ is the fundamental eigenvector, the n results for the eigenvalue will of course all be equal to the exact fundamental eigenvalue.

Another method for writing a single equation of motion in p involves the requirement that a weighted sum of the errors $\{e\}$ be zero. In the most important method of this description, Galerkin's method, each of the terms in $\{e\}$ is weighted by the corresponding term in the shape $\{\gamma\}$ We can write that

$$\lfloor \gamma_1, \gamma_2, \ldots, \gamma_n \rfloor \begin{Bmatrix} e_1 \\ e_2 \\ \cdot \\ e_n \end{Bmatrix} = 0 \tag{6.80}$$

If we interpret $\{\gamma\}$ as a displacement vector and $\{e\}$ as a force vector, we can say that Galerkin's method requires the error in the forces to be orthogonal to the assumed shape of the displacements. Substitution of Eq. (6 77) into Eq. (6.80) leads to the equation of motion in p, given by

$$-\lfloor \gamma \rfloor [m] \{\gamma\} \ddot{p} - \lfloor \gamma \rfloor [k] \{\gamma\} p + \lfloor \gamma \rfloor \{F\}_{ex} = 0 \tag{6.81}$$

In view of Eqs. (6 52) and (6 55), this equation is seen to be identical with Eq. (6.56) Thus Galerkin's method gives the same result as Rayleigh's method. The approximation for the fundamental eigenvalue is given by Eq. (6.57).

EXAMPLE *6.9*

In Example 6 7, an estimate was obtained for the fundamental torsional frequency of the system of Fig. 6-4, using Rayleigh's method. Let us obtain new estimates using methods based on the equations of motion. Using the

same shape as was used in Example 6.7, we can write the equations of transformation as

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} \frac{7}{16} \\ \frac{3}{4} \\ \frac{15}{16} \\ 1 \end{Bmatrix} p$$

The equations of motion for a free vibration of the system are given by

$$-\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & \frac{I}{2} \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \end{Bmatrix} - \begin{bmatrix} 2k & -k & 0 & 0 \\ -k & 2k & -k & 0 \\ 0 & -k & 2k & -k \\ 0 & 0 & -k & k \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Substitution of the equations of transformation into the equations of motion leads to

$$-\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & \frac{I}{2} \end{bmatrix} \begin{Bmatrix} \frac{7}{16} \\ \frac{3}{4} \\ \frac{15}{16} \\ 1 \end{Bmatrix} \ddot{p} - \begin{bmatrix} 2k & -k & 0 & 0 \\ -k & 2k & -k & 0 \\ 0 & -k & 2k & -k \\ 0 & 0 & -k & k \end{bmatrix} \begin{Bmatrix} \frac{7}{16} \\ \frac{3}{4} \\ \frac{15}{16} \\ 1 \end{Bmatrix} p = \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{Bmatrix}$$

or to

$$\begin{aligned} -\tfrac{7}{16} I\ddot{p} - \tfrac{1}{8} kp &= e_1 \\ -\tfrac{3}{4} I\ddot{p} - \tfrac{1}{8} kp &= e_2 \\ -\tfrac{15}{16} I\ddot{p} - \tfrac{1}{8} kp &= e_3 \\ -\tfrac{1}{2} I\ddot{p} - \tfrac{1}{16} kp &= e_4 \end{aligned}$$

Using the method of collocation, if we can set the errors e_1, e_2, e_3, e_4 to zero in turn, leading to the four estimates of the fundamental eigenvalue, given by

$$\begin{aligned} \omega^2 &= \frac{2}{7}\frac{k}{I} = 0.286\,\frac{k}{I} \\ &= \frac{1}{6}\frac{k}{I} = 0.167\,\frac{k}{I} \\ &= \frac{2}{15}\frac{k}{I} = 0.133\,\frac{k}{I} \\ &= \frac{1}{8}\frac{k}{I} = 0.125\,\frac{k}{I} \end{aligned}$$

Note that the error can be quite large. Using Galerkin's method, we can write

$$-\left\lfloor \tfrac{7}{16}\ \tfrac{3}{4}\ \tfrac{15}{16}\ 1 \right\rfloor \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & \frac{I}{2} \end{bmatrix} \begin{Bmatrix} \frac{7}{16} \\ \frac{3}{4} \\ \frac{15}{16} \\ 1 \end{Bmatrix} \ddot{p}$$

$$- \left\lfloor \tfrac{7}{16}\ \tfrac{3}{4}\ \tfrac{15}{16}\ 1 \right\rfloor \begin{bmatrix} 2k & -k & 0 & 0 \\ -k & 2k & -k & 0 \\ 0 & -k & 2k & -k \\ 0 & 0 & -k & k \end{bmatrix} \begin{Bmatrix} \frac{7}{16} \\ \frac{3}{4} \\ \frac{15}{16} \\ 1 \end{Bmatrix} p = 0$$

which simplifies to

$$-2.133 I\ddot{p} - 0.3281 kp = 0$$

Then the estimate for the fundamental eigenvalue is given by

$$\omega^2 = \frac{0.3281k}{2.133I} = 0.1538\frac{k}{I}$$

This result compares with that given by Rayleigh's method in Example 6 7 as it should. Evidently Galerkin's method gives a much better result than we can expect from the method of collocation.

(*b*) *Reduction to Several Generalized Coordinates*

Let us represent the motion of the system with more than one coordinate as given by the equations of transformation, Eq (6 58). Substitution of Eq. (6 58) into the equations of motion, Eq (5.24), leads to

$$\begin{Bmatrix} \sum F_1 \\ \vdots \\ \sum F_n \end{Bmatrix} = -\begin{bmatrix} m_{11} & \cdots & m_{1n} \\ & & \\ m_{n1} & & m_{nn} \end{bmatrix} \begin{bmatrix} \gamma_{11} & & \gamma_{1k} \\ & & \\ \gamma_{n1} & \cdots & \gamma_{nk} \end{bmatrix} \begin{Bmatrix} \ddot{p}_1 \\ \\ \ddot{p}_k \end{Bmatrix}$$

$$-\begin{bmatrix} k_{11} & & k_{1n} \\ \vdots & & \\ k_{n1} & & k_{nn} \end{bmatrix} \begin{bmatrix} \gamma_{11} & & \gamma_{1k} \\ & & \\ \gamma_{n1} & & \gamma_{nk} \end{bmatrix} \begin{Bmatrix} p_1 \\ \\ p_k \end{Bmatrix} + \begin{Bmatrix} F_1 \\ \\ F_n \end{Bmatrix}_{ex} = \begin{Bmatrix} e_1 \\ \\ e_n \end{Bmatrix} \tag{6 82}$$

Since there are k generalized coordinates $\{p\}$, we will need to reduce the number of equations of motion to k Using the concept of collocation, we can write these equations by requiring that k of the errors $\{e\}$ be zero If, instead, we use Galerkin's method, we will weight the errors $\{e\}$ by each of the shapes $\{\gamma\}$ and set the weighted errors separately to zero We can write the result as

$$\begin{bmatrix} \gamma_{11} & & \gamma_{n1} \\ & & \\ \gamma_{1k} & \cdots & \gamma_{nk} \end{bmatrix} \begin{Bmatrix} e_1 \\ \vdots \\ e_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ \\ 0 \end{Bmatrix} \tag{6 83}$$

It is common practice to describe the coupled bending and torsional vibrations of a wing in terms of the uncoupled bending and torsional normal modes. The normal modes for uncoupled bending vibration are obtained for the wing constrained against twisting. Similarly, the normal modes for uncoupled torsional vibration are obtained for the wing constrained against bending. Suppose in this example we know the natural frequencies and normal mode shapes of the fundamental modes of uncoupled bending and torsional vibrations, given by

$$\omega_1^{x2} = 2.92 \frac{EI}{ml^3}$$

$$\{\phi\}_1^x = \begin{Bmatrix} 0.096 \\ 0.336 \\ 0.654 \\ 1 \end{Bmatrix}$$

$$\omega_1^{\theta 2} = 0.609 \frac{GJ}{mr^2 l}$$

$$\{\phi\}_1^\theta = \begin{Bmatrix} 0.383 \\ 0.707 \\ 0.924 \\ 1 \end{Bmatrix}$$

The superscripts x and θ indicate bending and twisting respectively. Let us approximate the coupled bending and torsional vibrations of the wing with these two modes.

Referring to the free-body diagram of Fig. 6-9, we can write the generalized inertia forces as

$$\begin{Bmatrix} F_1^x \\ F_2^x \\ F_3^x \\ F_4^x \\ \hline F_1^\theta \\ F_2^\theta \\ F_3^\theta \\ F_4^\theta \end{Bmatrix}_{in} = -\left[\begin{array}{cccc:cccc} m & 0 & 0 & 0 & -me & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & -me & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 & -me & 0 \\ 0 & 0 & 0 & \frac{m}{2} & 0 & 0 & 0 & -\frac{me}{2} \\ \hdashline -me & 0 & 0 & 0 & mr^2 & 0 & 0 & 0 \\ 0 & -me & 0 & 0 & 0 & mr^2 & 0 & 0 \\ 0 & 0 & -me & 0 & 0 & 0 & mr^2 & 0 \\ 0 & 0 & 0 & -\frac{me}{2} & 0 & 0 & 0 & \frac{mr^2}{2} \end{array}\right] \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \\ \hline \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \end{Bmatrix}$$

or, more compactly, as

$$\left\{\begin{matrix} F^x \\ \hdashline F^\theta \end{matrix}\right\}_{in} = -[m]\left\{\begin{matrix} \ddot{x} \\ \hdashline \ddot{\theta} \end{matrix}\right\}$$

with

$$[m] = \left[\begin{array}{c:c} [m]^{xx} & [m]^{x\theta} \\ \hdashline [m]^{\theta x} & [m]^{\theta\theta} \end{array}\right]$$

Similarly we can write the generalized elastic forces as

$$\left\{\begin{matrix} F^x \\ \hdashline F^\theta \end{matrix}\right\}_{el} = -[k]\left\{\begin{matrix} x \\ \hdashline \theta \end{matrix}\right\}$$

with

$$[k] = \left[\begin{array}{c:c} [k]^{xx} & [0] \\ \hdashline [0] & [k]^{\theta\theta} \end{array}\right]$$

The zero matrices result from the fact that there is no elastic coupling between translation and rotation. The equations of motion for a free vibration of the wing are given by

$$\left\{\begin{matrix} \sum F^x \\ \hdashline \sum F^\theta \end{matrix}\right\} = -[m]\left\{\begin{matrix} \ddot{x} \\ \hdashline \ddot{\theta} \end{matrix}\right\} - [k]\left\{\begin{matrix} x \\ \hdashline \theta \end{matrix}\right\} = \{0\}$$

We can write the equations of transformation as

$$\left\{\begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \hdashline \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{matrix}\right\} = \left[\begin{array}{c:c} 0.096e & 0 \\ 0.336e & 0 \\ 0.654e & 0 \\ e & 0 \\ \hdashline 0 & 0.383 \\ 0 & 0.707 \\ 0 & 0.924 \\ 0 & 1 \end{array}\right]\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

Thus p_1 represents motion in the fundamental uncoupled bending mode an p_2 represents motion in the fundamental uncoupled torsional mode. The firs column was multiplied by the length e arbitrarily to make the coordinate p nondimensional. In a more compact form, the equations of transformatior are given by

$$\left\{\begin{matrix} x \\ \hdashline \theta \end{matrix}\right\} = [\gamma]\{p\}$$

with

$$[\gamma] = \left[\begin{array}{c:c} e\{\phi\}_1^x & \{0\} \\ \hdashline \{0\} & \{\phi\}_1^\theta \end{array}\right]$$

If we substitute the equations of transformation into the equations of motion and make use of Galerkin's method, the resulting equations of motion in $\{p\}$ are given by

$$-[\gamma]'[m][\gamma]\{\ddot{p}\} - [\gamma]'[k][\gamma]\{p\} = \{0\}$$

We can write the inertia matrix as

$$\begin{aligned}[M] &= [\gamma]'[m][\gamma] \\ &= \left[\begin{array}{c:c} e\lfloor\phi\rfloor_1^x & \lfloor 0\rfloor \\ \hdashline \lfloor 0\rfloor & \lfloor\phi\rfloor_1^\theta \end{array}\right] \left[\begin{array}{c:c} [m]^{xx} & [m]^{x\theta} \\ \hdashline [m]^{\theta x} & [m]^{\theta\theta} \end{array}\right] \left[\begin{array}{c:c} e\{\phi\}_1^x & \{0\} \\ \hdashline \{0\} & \{\phi\}_1^\theta \end{array}\right]\end{aligned}$$

or, by performing the multiplications, as

$$[M] = \left[\begin{array}{c:c} e^2\lfloor\phi\rfloor_1^x[m]^{xx}\{\phi\}_1^x & e\lfloor\phi\rfloor_1^x[m]^{x\theta}\{\phi\}_1^\theta \\ \hdashline e\lfloor\phi\rfloor_1^\theta[m]^{\theta x}\{\phi\}_1^x & \lfloor\phi\rfloor_1^\theta[m]^{\theta\theta}\{\phi\}_1^\theta \end{array}\right]$$

The individual generalized masses are given by

$$\begin{aligned}M_{11} &= e^2\lfloor\phi\rfloor_1^x[m]^{xx}\{\phi\}_1^x \\ &= e^2\lfloor 0.096\ 0.336\ 0.654\ 1\rfloor \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & \frac{m}{2} \end{bmatrix} \begin{Bmatrix} 0.096 \\ 0.336 \\ 0.654 \\ 1 \end{Bmatrix} \\ &= 1.050me^2\end{aligned}$$

$$\begin{aligned}M_{12} &= e\lfloor\phi\rfloor_1^x[m]^{x\theta}\{\phi\}_1^\theta \\ &= e\lfloor 0.096\ 0.336\ 0.654\ 1\rfloor \begin{bmatrix} -me & 0 & 0 & 0 \\ 0 & -me & 0 & 0 \\ 0 & 0 & -me & 0 \\ 0 & 0 & 0 & -\frac{me}{2} \end{bmatrix} \begin{Bmatrix} 0.383 \\ 0.707 \\ 0.924 \\ 1 \end{Bmatrix} \\ &= -1.380me^2\end{aligned}$$

$$\begin{aligned}M_{21} &= e\lfloor\phi\rfloor_1^\theta[m]^{\theta x}\{\phi\}_1^x \\ &= M_{12}\end{aligned}$$

$$\begin{aligned}M_{22} &= \lfloor\phi\rfloor_1^\theta[m]^{\theta\theta}\{\phi\}_1^\theta \\ &= \lfloor 0.383\ 0.707\ 0.924\ 1\rfloor \begin{bmatrix} mr^2 & 0 & 0 & 0 \\ 0 & mr^2 & 0 & 0 \\ 0 & 0 & mr^2 & 0 \\ 0 & 0 & 0 & \frac{mr^2}{2} \end{bmatrix} \begin{Bmatrix} 0.383 \\ 0.707 \\ 0.924 \\ 1 \end{Bmatrix} \\ &= 2.000mr^2\end{aligned}$$

Similarly, we can write the stiffness matrix as

$$\begin{aligned}[K] &= [\gamma]'[k][\gamma] \\ &= \left[\begin{array}{c:c} e\lfloor\phi\rfloor_1^x & \lfloor 0\rfloor \\ \hdashline \lfloor 0\rfloor & \lfloor\phi\rfloor_1^\theta \end{array}\right]\left[\begin{array}{c:c} [k]^{xx} & [0] \\ \hdashline [0] & [k]^{\theta\theta} \end{array}\right]\left[\begin{array}{c:c} e\{\phi\}_1^x & \{0\} \\ \hdashline \{0\} & \{\phi\}_1^\theta \end{array}\right] \\ &= \left[\begin{array}{c:c} e^2\lfloor\phi\rfloor_1^x[k]^{xx}\{\phi\}_1^x & 0 \\ \hdashline 0 & \lfloor\phi\rfloor_1^\theta[k]^{\theta\theta}\{\phi\}_1^\theta \end{array}\right]\end{aligned}$$

The generalized stiffness K_{11} is just the stiffness for uncoupled bending vibrations in the fundamental mode. From Eq. (5.57) we can write

$$K_{11} = M_{11}\omega_1^{x^2} = 1.050me^2\omega_1^{x^2}$$

The generalized stiffness K_{22} is the stiffness for uncoupled torsional vibrations in the fundamental mode. It follows that

$$K_{22} = M_{22}\omega_1^{\theta^2} = 2.000mr^2\omega_1^{\theta^2}$$

The equations of motion in $\{p\}$ are

$$-\begin{bmatrix} 1.050me^2 & -1.380me^2 \\ -1.380me^2 & 2.000mr^2 \end{bmatrix}\begin{Bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{Bmatrix} - \begin{bmatrix} 1.050me^2\omega_1^{x^2} & 0 \\ 0 & 2.000mr^2\omega_1^{\theta^2} \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} \quad \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Solution of this pair of equations will yield an estimate for the fundamental natural frequency and normal mode shape for the coupled motion.

6.6 Dunkerley's Equation

For systems in which the natural frequencies are well separated, we can obtain an approximation for the fundamental natural frequency using Dunkerley's equation. This method is particularly useful for the case of flexural vibrations of beams.

Let us rewrite Eq. (6.1) in the form

$$\left[\frac{1}{\omega^2}[I] - [D]\right]\{A\} = \{0\} \tag{6.85}$$

in which $[I]$ is the unit matrix and

$$[D] = [k]^{-1}[m] = [C][m] \tag{6.86}$$

The characteristic equation is

$$\left|\frac{1}{\omega^2}[I] - [D]\right| = 0 \tag{6.87}$$

If we expand this equation, the first two terms are given by

$$\left(\frac{1}{\omega^2}\right)^n - (D_{11} + D_{22} + \cdots + D_{nn})\left(\frac{1}{\omega^2}\right)^{n-1} + \cdots = 0 \qquad (6.88)$$

in which the second coefficient is the sum of the terms on the diagonal of $[D]$, referred to as the trace of $[D]$. We can write the factored form of the characteristic equation as

$$\left(\frac{1}{\omega^2} - \frac{1}{\omega_1^2}\right)\left(\frac{1}{\omega^2} - \frac{1}{\omega_2^2}\right) \cdots \left(\frac{1}{\omega^2} - \frac{1}{\omega_n^2}\right) = 0 \qquad (6.89)$$

or as

$$\left(\frac{1}{\omega^2}\right)^n + \left[\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \cdots + \frac{1}{\omega_n^2}\right]\left(\frac{1}{\omega^2}\right)^{n-1} + \cdots = 0 \qquad (6.90)$$

Comparison of Eqs. (6.88) and (6.90) leads to the statement

$$\begin{aligned}\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \cdots + \frac{1}{\omega_n^2} &= D_{11} + D_{22} + \cdots + D_{nn} \\ &= \text{trace } [D]\end{aligned} \qquad (6.91)$$

Having assumed that the higher natural frequencies are much larger than the fundamental frequency, we can ignore all the terms on the left-hand side of the equation except the first, leading to the approximation

$$\frac{1}{\omega_1^2} \approx \text{trace } [D] \qquad (6.92)$$

From the assumption made, it is evident that the fundamental natural frequency obtained from this equation will always be lower than the exact value.

If there is no inertial coupling, we can give the right-hand side of Eq. (6.92) a particular physical meaning. For this case the terms on the diagonal of $[D]$ can be written in the form

$$D_{ii} = C_{ii}m_i \qquad (6.93)$$

If we set all the masses to zero except m_i, the system becomes a single-degree-of-freedom system having a natural frequency given by

$$\frac{1}{\omega_{ii}^2} = C_{ii}m_i \qquad (6.94)$$

Substitution of Eqs. (6.93) and (6.94) into Eq. (6.92) leads to Dunkerley's equation, written as

$$\frac{1}{\omega_1^2} \approx \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} + \cdots + \frac{1}{\omega_{nn}^2} \qquad (6.95)$$

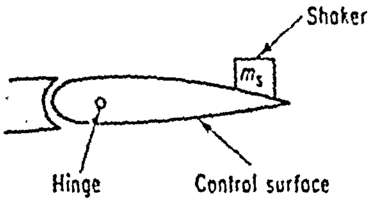


Fig. 6-10 *Mechanical shaker mounted on a control surface.*

EXAMPLE 6.11

A mechanical shaker of mass m_s is mounted on an airplane control surface as shown in Fig. 6-10 for the purpose of obtaining the resonant frequency. The measured frequency ω_s is that for the system with the added shaker mass. If we measure the flexibility influence coefficient C_{ss} for the mounting point of the shaker, we can write Dunkerley's equation as

$$\frac{1}{\omega_s^2} = \frac{1}{\omega_c^2} + C_{ss}m_s$$

The natural frequency ω_c for the control surface without the shaker mass is given approximately by

$$\frac{1}{\omega_c^2} = \frac{1}{\omega_s^2} - C_{ss}m_s$$

EXAMPLE 6.12

Let us obtain an estimate for the fundamental natural frequency of the lumped mass cantilever beam of Fig. 6-8. The influence coefficients associated with x_1 and x_2 are

$$C_{11} = \frac{l^3}{3EI}$$
$$C_{22} = \frac{8l^3}{3EI}$$

Then the natural frequencies of the beam having only the first mass or only the second mass are given by

$$\frac{1}{\omega_{11}^2} = C_{11}m = \frac{ml^3}{3EI}$$
$$\frac{1}{\omega_{22}^2} = C_{22}m = \frac{8ml^3}{3EI}$$

From Dunkerley's equation, written as

$$\frac{1}{\omega_1^2} \approx \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} = \frac{ml^3}{3EI} + \frac{8ml^3}{3EI}$$
$$\approx \frac{3ml^3}{EI}$$

we obtain the approximation for the fundamental eigenvalue, given by

$$\omega_1^2 \approx 0.333 \frac{EI}{ml^3}$$

This result compares with the exact value of $\omega_1^2 = 0.341 \frac{EI}{ml^3}$ given in Example 6.6.

EXAMPLE 6.13

The method just outlined will not be useful if the natural frequencies are not well separated. For example, this is the case for the system of Fig. 5-1. From Example 5.4, the three eigenvalues for the system are

$$\omega_1^2 = 2.54 \frac{k}{m}$$
$$\omega_2^2 = 6 \frac{k}{m}$$
$$\omega_3^2 = 9.46 \frac{k}{m}$$

Making use of Example 6.1, we can write the matrix $[D]$ of Eq. (6.86) as

$$[D] = [C][m]$$
$$= \frac{m}{12k} \begin{bmatrix} 6 & -l & l \\ \frac{3}{l} & 1 & 1 \\ -\frac{3}{l} & 1 & 1 \end{bmatrix}$$

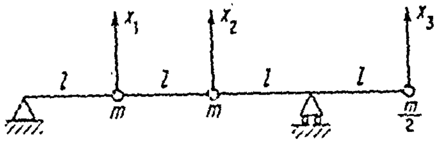
From Eq. (6.92), the approximation for the fundamental eigenvalue of the system is given by

$$\frac{1}{\omega_1^2} \approx \text{trace } [D]$$
$$\approx \frac{m}{12k}(6 + 1 + 1) = \frac{2m}{3k}$$

Evidently the result $\omega_1^2 \approx 1.5\frac{k}{m}$ is not a good estimate for the fundamental eigenvalue, given exactly by $\omega_1^2 = 2.54 \frac{k}{m}$.

Problems

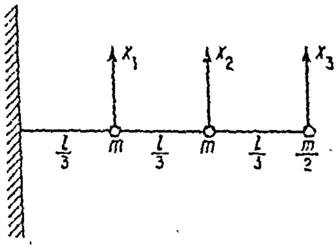
6-1 A uniform beam of flexural rigidity EI, supported as shown, carries two particles of mass m and one of mass $m/2$. Write the equations of motion for the flexural vibrations of the system. Obtain the lowest natural frequency and mode shape, using the method of iteration.



Prob. 6-1

6-2 For the triple pendulum of Prob. 5-5, obtain the lowest and highest natural frequencies and mode shapes, using the method of iteration. Determine the intermediate normal mode shape using the orthogonality relations.

6-3 Consider the lumped-mass approximation for a uniform cantilever beam as shown. If the flexural rigidity is given by EI, write the flexibility influence coefficients associated with the coordinates x_1, x_2, x_3. Write the equations of motion and iterate to obtain the lowest natural frequency and mode shape.

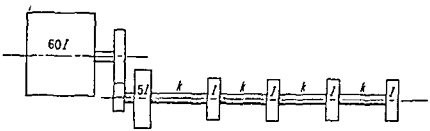


Prob. 6-3

6-4 Sweep the fundamental mode out of the equations of motion for the system of Prob. 6-1 and iterate to obtain the second natural frequency and mode shape. Determine the highest normal mode shape, using the orthogonality relations.

6-5 A uniform shaft carries four equally spaced disks of axial moment of inertia I. The left-hand end of the shaft carries a flywheel and pinion with a combined axial moment of inertia $5I$. The pinion drives a gear connected to a rotor. The gear and rotor have an axial moment of inertia $60I$. The gear ratio is 5 to 1. As shown, the shaft segments are represented by the torsional

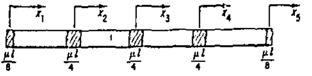
stiffness k. Note that there is a rigid-body degree of freedom. Determine the natural frequency and mode shape for the first elastic mode, using the method of iteration.



Prob. 6-5

6-6 For the system of Prob 6-5, sweep out the first elastic mode and iterate to obtain the natural frequency and mode shape of the second elastic mode.

6-7 A uniform bar of length l and mass per unit length μ is approximated by the massless bar carrying five equally spaced mass lumps as shown The stiffness of each of the segments of the bar in an axial displacement is given by $k = \dfrac{4EA}{l}$ in which A is the cross-sectional area and E is Young's modulus. The ends of the bar are free Iterate the equations of motion to obtain the natural frequency and mode shape for the first elastic mode.

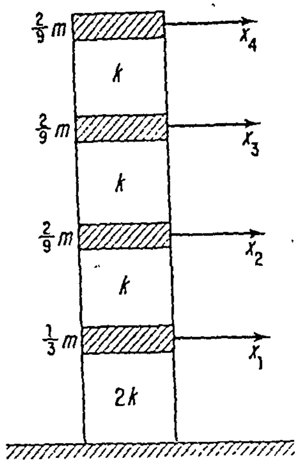


Prob 6-7

6-8 Use Holzer's method to obtain the three natural frequencies and mode shapes of the system of Prob. 5-4 Check your results, using the orthogonality relations.

6-9 Determine the natural frequency and mode shape of the first elastic mode of the system of Prob. 6-5, using Holzer's method.

6-10 A five-story building is idealized as a four-degree-of-freedom, lumped-mass system as shown The shearing stiffness between floors is indicated. Obtain the lowest natural frequency and mode shape for the idealized building, using Holzer's method.



Prob. 6-10

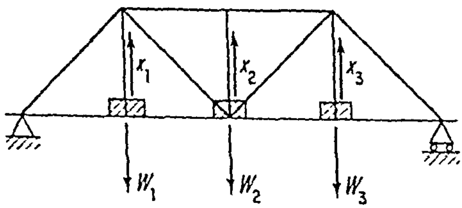
6-11 For the system of Prob. 6-7, determine the natural frequency and mode shape of the first elastic mode, using Holzer's method.

6-12 Using the method of transfer matrices, write the characteristic equation for the system of Prob. 5-4. Obtain the three natural frequencies. Determine the state vectors for the fundamental mode, normalizing by setting the maximum displacement to unity.

6-13 Write the characteristic equation for the torsional system of Prob. 5-2 using the method of transfer matrices. Determine the natural frequencies of the system.

6-14 For the symmetric mode of flexural vibration of the idealized airplane of Prob. 5-13, determine the characteristic equation using the method of transfer matrices. Obtain the natural frequencies of the system. Write the state vectors for the lowest elastic mode, normalizing by setting the wing-tip displacement to unity.

6-15 A bridge structure is idealized as a massless truss system supporting three weights as shown. The weights and the flexibility influence coefficients



Prob. 6-15

associated with the coordinates x_1, x_2, x_3 are given by

$$W_1 = 6 \text{ tons} \qquad W_2 = 10 \text{ tons} \qquad W_3 = 8 \text{ tons} \qquad [C] = \begin{bmatrix} 0.40 & 0.40 & 0.32 \\ 0.40 & 0.48 & 0.40 \\ 0.32 & 0.40 & 0.40 \end{bmatrix} \text{in./ton}$$

Approximating the fundamental mode shape by

$$\{y\} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

obtain an estimate for the fundamental natural frequency, using Rayleigh's method. Compare the results with those obtained by using the method of iteration.

6-16 Let us approximate the fundamental mode shape of the cantilever beam of Prob. 6-3 by the shape taken by the beam under the static gravity loading, given by

$$\{y\} = \begin{Bmatrix} 0.175 \\ 0.556 \\ 1 \end{Bmatrix}$$

Obtain an estimate for the fundamental natural frequency using Rayleigh's method.

6-17 For the triple pendulum of Prob. 5-5, let us approximate the fundamental mode shape in terms of the angles $\theta_1, \theta_2, \theta_3$ by

$$\{y\} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

This is the shape which results if equal transverse forces are applied to the three masses. Using Rayleigh's method, obtain an estimate for the fundamental natural frequency. Compare the result with that of Prob. 6-2.

6-18 Let us represent the motion of the triple pendulum of Prob. 5-5 with two coordinates whose shapes in terms of the angles $\theta_1, \theta_2, \theta_3$, are given by

$$[y] = \begin{bmatrix} 1 & \frac{1}{3} \\ 1 & \frac{2}{3} \\ 1 & 1 \end{bmatrix}$$

Using the energy approach, write the equations of motion in the new coordinates. Obtain the natural frequencies and mode shapes for the approximated motion. Compare the results for the fundamental mode with those of Prob. 6-17.

6-19 For the system of Prob. 6-10, write the equations of motion in the two coordinates whose shapes are given by

$$[\gamma] = \begin{bmatrix} 0.5 & -0.8 \\ 0.8 & -0.8 \\ 0.9 & 0.2 \\ 1 & 1 \end{bmatrix}$$

The two shapes selected represent the first two mode shapes of a similar building. Use the energy approach. Obtain the natural frequencies and mode shapes and compare the fundamental mode with the exact result of Prob. 6-10.

6-20 In Prob. 6-16, use the method of collocation to obtain three different estimates for the fundamental natural frequency.

6-21 Use the method of collocation in Prob. 6-17 to obtain three different estimates for the fundamental natural frequency.

6-22 For the system of Prob. 6-1, write the equation of motion in the single coordinate whose shape is given by

$$\{\gamma\} = \begin{Bmatrix} -0.633 \\ -0.667 \\ 1 \end{Bmatrix}$$

This is the shape which results if transverse forces proportional to the mass of the particles are applied to the particles. The direction of the third force is opposite to that for the first two forces. Use Galerkin's method in writing the equation. Obtain the estimate for the fundamental natural frequency and compare it with the result of Prob. 6-1.

6-23 Consider the cantilever beam of Prob. 6-3. For the two coordinates with shapes given by

$$[\gamma] = \begin{bmatrix} \frac{1}{9} & \frac{1}{27} \\ \frac{4}{9} & \frac{8}{27} \\ 1 & 1 \end{bmatrix}$$

write the equations of motion, using Galerkin's method. Determine the natural frequencies and mode shapes for the approximated motion. Compare the results for the fundamental mode with those of Prob. 6-3.

6-24 Use Dunkerley's equation to obtain an estimate for the fundamental natural frequency for the system of Prob. 6-1.

6-25 Use Dunkerley's equation to obtain an estimate for the fundamental natural frequency for the system of Prob. 6-10.

CHAPTER 7

Forced Vibrations of Lumped-Mass Systems

7.1 Solution for the Motion and the Resulting Elastic Forces

(a) *The Mode-Displacement Method*

For a general n-degree-of-freedom system we can write the equations of motion for a forced vibration in the coordinates $\{x\}$ as

$$\{\sum F\} = -[m]\{\ddot{x}\} - [k]\{x\} + \{F\}_{ex} = \{0\} \tag{7.1}$$

Unless the coordinates are the normal coordinates, the equations will in general be coupled inertially and elastically. We can solve the coupled equations directly with an analog computer, or with a digital computer using numerical methods. However, neither of these are the usual practice; generally we will find it useful to transform the equations of motion to the normal coordinates. The equations of transformation are given by

$$\{x\} = [\phi]\{q\} \tag{7.2}$$

in which the columns of $[\phi]$ are the normal mode shapes. From Eq. (5.62), we can write the equations of motion in the normal coordinates $\{q\}$ as

$$\sum Q_i = -M_i\ddot{q}_i - K_i q_i + Q_{i\,ex} = 0 \qquad i = 1, \ldots, n \tag{7.3}$$

These uncoupled equations can be solved using the methods outlined in Chap. 3 for single-degree-of-freedom systems. Since the stiffness of the normal modes increases with the mode number, we will often find that the response in the higher modes is small. As a result, we may only need to determine the response in the first few normal coordinates.

Although the displacements of a system in a forced vibration will be of interest, we will more often be concerned about the internal elastic forces

6-19 For the system of Prob. 6-10, write the equations of motion in the two coordinates whose shapes are given by

$$[y] = \begin{bmatrix} 0.5 & -0.8 \\ 0.8 & -0.8 \\ 0.9 & 0.2 \\ 1 & 1 \end{bmatrix}$$

The two shapes selected represent the first two mode shapes of a similar building. Use the energy approach. Obtain the natural frequencies and mode shapes and compare the fundamental mode with the exact result of Prob. 6-10.

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6-23 Consider the cantilever beam of Prob. 6-3. For the two coordinates with shapes given by

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write the equations of motion, using Galerkin's method. Determine the natural frequencies and mode shapes for the approximated motion. Compare the results for the fundamental mode with those of Prob. 6-3.

6-24 Use Dunkerley's equation to obtain an estimate for the fundamental natural frequency for the system of Prob. 6-1.

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This is the shape which results if transverse forces proportional to the mass of the particles are applied to the particles. The direction of the third force is opposite to that for the first two forces. Use Galerkin's method in writing the equation. Obtain the estimate for the fundamental natural frequency and compare it with the result of Prob. 6-1.

6-23 Consider the cantilever beam of Prob. 6-3. For the two coordinates with shapes given by

$$[\gamma] = \begin{bmatrix} \frac{1}{9} & \frac{1}{27} \\ \frac{4}{9} & \frac{8}{27} \\ 1 & 1 \end{bmatrix}$$

write the equations of motion, using Galerkin's method. Determine the natural frequencies and mode shapes for the approximated motion. Compare the results for the fundamental mode with those of Prob. 6-3.

6-24 Use Dunkerley's equation to obtain an estimate for the fundamental natural frequency for the system of Prob. 6-1.

6-25 Use Dunkerley's equation to obtain an estimate for the fundamental natural frequency for the system of Prob. 6-10.

Forced Vibrations of Lumped-Mass Systems

7.1 Solution for the Motion and the Resulting Elastic Forces

(a) *The Mode-Displacement Method*

For a general n-degree-of-freedom system we can write the equations of motion for a forced vibration in the coordinates $\{x\}$ as

$$\{\sum F\} = -[m]\{\ddot{x}\} - [k]\{x\} + \{F\}_{ex} = \{0\} \tag{7.1}$$

Unless the coordinates are the normal coordinates, the equations will in general be coupled inertially and elastically. We can solve the coupled equations directly with an analog computer, or with a digital computer using numerical methods. However, neither of these are the usual practice; *generally we will find it useful to transform the equations of motion to the* normal coordinates. The equations of transformation are given by

$$\{x\} = [\phi]\{q\} \tag{7.2}$$

in which the columns of $[\phi]$ are the normal mode shapes. From Eq. (5.62), we can write the equations of motion in the normal coordinates $\{q\}$ as

$$\sum Q_i = -M_i\ddot{q}_i - K_iq_i + Q_{i,ex} = 0 \qquad i = 1, \ldots, n \tag{7.3}$$

These uncoupled equations can be solved using the methods outlined in Chap. 3 for single-degree-of-freedom systems. Since the stiffness of the normal modes increases with the mode number, we will often find that the response in the higher modes is small. As a result, we may only need to determine the response in the first few normal coordinates.

Although the displacements of a system in a forced vibration will be of interest, we will more often be concerned about the internal elastic forces

or stresses. Consider the problem of determining the internal elastic force at a selected point of the system. The applied forces required to give the system a unit displacement in the ith normal mode are

$$\{F\} = [k]\{\phi\}_i \tag{7.4}$$

Then the internal elastic force at the selected point per unit displacement of the ith normal mode is just that which would result from the applied forces of Eq. (7.4). Making reference to Eq. (5.40), we can write Eq. (7.4) in the alternate form

$$\{F\} = \omega_i^2[m]\{\phi\}_i \tag{7.5}$$

The right-hand side of Eq. (7.5) represents the inertia forces resulting from a free vibration of unit amplitude in the ith normal mode. Let us identify the internal elastic force at the selected point per unit displacement of the ith normal mode as l_i. Then we can write

$$L = \lfloor l_1\ l_2 \cdots l_n \rfloor \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{Bmatrix} \tag{7.6}$$

in which L represents the total internal elastic force at the selected point resulting from the forced vibration of the system. This method for determining the internal elastic forces is referred to as the mode-displacement method.

EXAMPLE 7.1

From Example 5.7 we can write the equations of motion for the forced vibration of the system of Fig. 5-6 in the normal coordinates as

$$\frac{4+2\sqrt{3}}{3}ml^2\ddot{q}_1 + \tfrac{4}{3}(3-\sqrt{3})kl^2q_1 = -\frac{\sqrt{3}}{3}Fl$$

$$\tfrac{2}{3}ml^2\ddot{q}_2 + 4kl^2q_2 = Fl$$

$$\frac{4-2\sqrt{3}}{3}ml^2\ddot{q}_3 + \tfrac{4}{3}(3+\sqrt{3})kl^2q_3 = \frac{\sqrt{3}}{3}Fl$$

Suppose the applied force is harmonic, as given by

$$F = F_0 \sin \omega t$$

If we are interested only in the steady-state motion, we can write the solution as

$$q_1 = \frac{\dfrac{1+\sqrt{3}}{8}\dfrac{F_o}{kl}}{1-\dfrac{\omega^2}{\omega_1^2}}\sin\omega t$$

$$q_2 = \frac{\dfrac{1}{4}\dfrac{Fo}{kl}}{1-\dfrac{\omega^2}{\omega_2^2}}\sin\omega t$$

$$q_3 = \frac{\dfrac{1-\sqrt{3}}{8}\dfrac{F_o}{kl}}{1-\dfrac{\omega^2}{\omega_3^2}}\sin\omega t$$

in which the natural frequencies from Example 5.4 are given by

$$\omega_1^2 = (6 - 2\sqrt{3})\frac{k}{m}$$

$$\omega_2^2 = 6\frac{k}{m}$$

$$\omega_3^2 = (6 + 2\sqrt{3})\frac{k}{m}$$

Referring to Fig. 5-6, suppose we are interested in the elastic force in the spring at the right-hand end of the system. The force L in the spring is related to the displacements x, θ, ϕ by

$$L = \lfloor 2k\ 0\ 2kl \rfloor \begin{Bmatrix} x \\ \theta \\ \phi \end{Bmatrix}$$

regarding tension as a positive force. Making use of the equation of transformation to the normal coordinates given in Example 5.7, we can write

$$L = \lfloor 2k\ 0\ 2kl \rfloor \begin{bmatrix} -\frac{1}{3}(3+\sqrt{3})l & 0 & -\frac{1}{3}(3-\sqrt{3})l \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

$$= \left\lfloor -\frac{2\sqrt{3}}{3}kl\ \ 2kl\ \ \frac{2\sqrt{3}}{3}kl \right\rfloor \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

Substitution of the solution for the steady-state forced vibration leads to the solution for the spring force, given by

$$L = \frac{F}{12}\left[\frac{3+\sqrt{3}}{1-\dfrac{\omega^2}{\omega_1^2}} + \frac{6}{1-\dfrac{\omega^2}{\omega_2^2}} + \frac{3-\sqrt{3}}{1-\dfrac{\omega^2}{\omega_3^2}}\right]\sin \omega t$$

EXAMPLE 7.2

Consider the lumped mass representation of the uniform beam shown in Fig. 7-1. The beam has been divided into three segments, each of length $\frac{1}{3}l$, and the mass of each has been lumped equally at the ends of the segment. As a result, if the mass of the beam is $3m$, two of the particles are of mass m and the end particle is of mass $m/2$. The flexural rigidity of the beam is given by EI. Suppose that we have obtained the normal mode shapes and natural frequencies for flexural vibrations of the beam, given by

$$\omega_1^2 = 3.73\frac{EI}{ml^3}$$

$$\omega_2^2 = 119\frac{EI}{ml^3}$$

$$\omega_3^2 = 738\frac{EI}{ml^3}$$

$$[\phi] = \begin{bmatrix} 0.162 & -0.730 & 2.23 \\ 0.540 & -0.707 & -1.59 \\ 1 & 1 & 1 \end{bmatrix}$$

As indicated in Fig. 7-1, a step force is applied to the end of the beam. The generalized masses for the normal coordinates are

$$[M] = \begin{bmatrix} 0.162 & 0.540 & 1 \\ -0.730 & -0.707 & 1 \\ 2.23 & -1.59 & 1 \end{bmatrix}\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \dfrac{m}{2} \end{bmatrix}\begin{bmatrix} 0.162 & -0.730 & 2.23 \\ 0.540 & -0.707 & -1.59 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.818m & 0 & 0 \\ 0 & 1.533m & 0 \\ 0 & 0 & 7.99m \end{bmatrix}$$

We can write the generalized stiffnesses as

$$K_1 = M_1\omega_1^2 = 3.05\frac{EI}{l^3}$$

$$K_2 = M_2\omega_2^2 = 182.4\frac{EI}{l^3}$$

$$K_3 = M_3\omega_3^2 = 5900\frac{EI}{l^3}$$

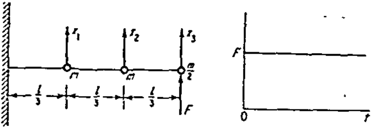


Fig. 7-1

The generalized external forces are

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix}_{ex} = \begin{bmatrix} 0.162 & 0.540 & 1 \\ -0.730 & -0.707 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ F \end{Bmatrix}$$

$$= \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} F$$

We can write the equations of motion in the normal coordinates as

$$0.818m\ddot{q}_1 + 3.05\frac{EI}{l^3}q_1 = F$$

$$1.533m\ddot{q}_2 + 182.4\frac{EI}{l^3}q_2 = F$$

$$7.99m\ddot{q}_3 + 5900\frac{EI}{l^3}q_3 = F$$

The time history of the applied force being a step, we can easily obtain the solution and can write

$$q_1 = 0.328\frac{Fl^3}{EI}(1 - \cos\omega_1 t)$$

$$q_2 = 0.00549\frac{Fl^3}{EI}(1 - \cos\omega_2 t)$$

$$q_3 = 0.000170\frac{Fl^3}{EI}(1 - \cos\omega_3 t)$$

Then the amplitude of the forced vibration of the end mass, taken as an example, is given by

$$x_3 = \frac{Fl^3}{EI}[1\ 1\ 1]\begin{Bmatrix} 0.328(1 - \cos\omega_1 t) \\ 0.00549(1 - \cos\omega_2 t) \\ 0.000170(1 - \cos\omega_3 t) \end{Bmatrix}$$

$$= \frac{Fl^3}{EI}(0.333 - 0.328\cos\omega_1 t - 0.00549\cos\omega_2 t - 0.000170\cos\omega_3 t)$$

Note the rapid convergence of the amplitude with the mode number. For engineering purposes, the responses in the second and third modes are negligible.

We are likely to be interested in the bending moment in the beam. From Eq. (7.5), the applied forces required to displace the system a unit amplitude in the fundamental mode are given by

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = 3.73\,\frac{EI}{l^3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{Bmatrix} 0.162 \\ 0.540 \\ 1 \end{Bmatrix}$$

$$= \frac{EI}{l^3}\begin{Bmatrix} 0.604 \\ 2.02 \\ 1.87 \end{Bmatrix}.$$

For the bending moment at the root of the beam, we can write

$$l_1 = \frac{EI}{l^3}\left\lfloor \tfrac{1}{3}l \;\; \tfrac{2}{3}l \;\; l \right\rfloor \begin{Bmatrix} 0.604 \\ 2.02 \\ 1.87 \end{Bmatrix}$$

$$= 3.41\,\frac{EI}{l^2}$$

in which l_1 represents the bending moment at the root per unit displacement of the fundamental mode. For the second and third modes

$$l_2 = -25.5\,\frac{EI}{l^2}$$

$$l_3 = 133\,\frac{EI}{l^2}$$

Then the solution for the bending moment at the root is given by

$$L = Fl\lfloor 3.41 \;\; -25.5 \;\; 133 \rfloor \begin{Bmatrix} 0.328\,(1 - \cos\omega_1 t) \\ 0.00549\,(1 - \cos\omega_2 t) \\ 0.000170\,(1 - \cos\omega_3 t) \end{Bmatrix}$$

$$= Fl(1.00 - 1.12\cos\omega_1 t + 0.140\cos\omega_2 t - 0.023\cos\omega_3 t)$$

Note that the solution for the bending moment is much less convergent with the mode number than is the solution for the displacement.

(*b*) *Reduction in Number of Coordinates*

Suppose, as outlined in Sec. 6.4 and 6.5, we have elected to approximate the motion with a reduced number of coordinates $\{p\}$, related to the original coordinates $\{x\}$ by

$$\{x\} = [\gamma]\{p\} \tag{7.7}$$

From Eq (6.64), we can write the equations of motion as

$$\{\Sigma P\} = -[M]\{\ddot{p}\} - [K]\{p\} + \{P\}_{ex} = \{0\} \tag{7.8}$$

These equations are in general coupled inertially and elastically. Although they can be solved directly, it is usually most convenient to transform the equations to a set of uncoupled equations. As outlined in Sec. 6.4, we can replace $\{p\}$ by an equal number of normal coordinates $\{q\}$. The normal coordinates are related to the original coordinates by

$$\{x\} = [\phi]\{q\} \tag{7.9}$$

From Eq. (6.76), we can write the equations of motion in the normal coordinates as

$$\sum Q_i = -M_i\ddot{q}_i - K_i q_i + Q_{i\,ex} = 0 \qquad i = 1, 2, \ldots, k \tag{7.10}$$

in which the reduced number of coordinates is k. These equations can be solved using the methods of Chap. 3. Since we began by reducing the number of coordinates and thus placing restrictions on the motion, the solution for the motion of the system will be approximate The internal elastic forces can be determined using the mode-displacement method outlined earlier Since the representation of the motion is approximate, the solution for the internal elastic forces will also be approximate.

EXAMPLE 7.3

In Example 6.8 the rotations $\{\theta\}$ of the disks of the four-degree-of-freedom torsional system of Fig 6-4 were approximately represented by the motion of the two selected coordinates $\{p\}$, given by

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{16} \\ \frac{1}{2} & \frac{1}{4} \\ \frac{3}{4} & \frac{9}{16} \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$$

From the results of Example 6.8, the relation between the rotations $\{\theta\}$ and the two normal coordinates $\{q\}$ is

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{bmatrix} 0.406 & -0.475 \\ 0.707 & -0.467 \\ 0.904 & 0.024 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

From an examination of the equations of motion, Eq. (7.3), we can identify the response of the normal coordinates to the inertia forces as

$$-M_i\ddot{q}_i - K_i q_{i,in} = 0 \qquad i = 1, 2, \ldots, n \tag{7.21}$$

or as

$$q_{i,in} = -\frac{M_i}{K_i}\ddot{q}_i \tag{7.22}$$

Making use of Eq. (5.57), which states that $K_i = M_i\omega_i^2$, we can write Eq. (7.22) in the alternate form

$$q_{i,in} = -\frac{\ddot{q}_i}{\omega_i^2} \qquad i = 1, 2, \ldots, n \tag{7.23}$$

Substitution of the individual responses, Eq. (7.23), into the second of Eqs. (7.18) leads to the solution for $\{x\}_{in}$, given by

$$\{x\}_{in} = -[\phi]\left[\diagdown \frac{1}{\omega^2} \diagdown\right]\{\ddot{q}\} \tag{7.24}$$

We can write an alternate solution for $\{x\}_{in}$ as

$$\{x\}_{in} = -[k]^{-1}[m][\phi]\{\ddot{q}\} \tag{7.25}$$

making use of Eq. (7.16). It will be more convenient usually to use Eq. (7.24). Since we are considering a reduced number of normal coordinates, the solution will be approximate. Combining the solutions for $\{x\}_{ex}$ and $\{x\}_{in}$, given by Eqs. (7.14) and (7.24), we can write the solution for the total response as

$$\{x\} = [k]^{-1}\{F\}_{ex} - [\phi]\left[\diagdown \frac{1}{\omega^2} \diagdown\right]\{\ddot{q}\} \tag{7.26}$$

Since the solution requires knowledge of the accelerations of the normal coordinates, the method is referred to as the mode acceleration method. The procedure as outlined is essentially that proposed by Williams (Ref. 18).

Let us compare the results of the mode acceleration method with those of the mode-displacement method. The total displacement response is given by the mode displacement method as

$$\{x\} = [\phi]\{q\} \tag{7.27}$$

or, using Eq. (7.12) as

$$\{x\} = [\phi]\{q\}_{ex} + [\phi]\{q\}_{in} \tag{7.28}$$

In view of Eq. (7.23), the second terms on the right-hand sides of Eqs. (7.26) and (7.28), representing $\{x\}_{in}$, are identical. Considering a reduced number of normal coordinates, the first terms on the right-hand side of Eqs. (7.26) and (7.28), representing $\{x\}_{ex}$, are different. They differ in that the term in

Eq. (7.26) is exact while the term in Eq. (7.28) is approximate. Thus the mode acceleration method represents an improvement over the mode-displacement method in that the response to the external forces has been made exact.

Usually the solution for the displacement response is sufficiently convergent with the mode number that it makes little difference whether we use the mode-displacement method or the mode-acceleration method. On the other hand, the solution for the internal elastic forces is generally much less convergent with the mode number and the choice of methods is likely to be important. Let us consider that the internal elastic force L at a selected point in the system consists of two parts, the part L_{ex} resulting from the external forces and the part L_{in} resulting from the inertia forces. Then

$$L = L_{ex} + L_{in} \tag{7.29}$$

We can determine the exact elastic force L_{ex} by applying the external forces and using the methods of mechanics of materials. From Eq. (7.6), we can indicate the solution for L_{in} by

$$L_{in} = \lfloor l \rfloor \{q\}_{in} \tag{7.30}$$

in which the elements of $\lfloor l \rfloor$ represent the internal elastic force per unit displacement of the normal coordinates. In view of Eq. (7.23), we can write that

$$L_{in} = -\lfloor l \rfloor \left[\diagdown \frac{1}{\omega^2} \diagdown \right] \{\ddot{q}\} \tag{7.31}$$

Then the solution for the internal elastic force L is given by

$$L = L_{ex} - \lfloor l \rfloor \left[\diagdown \frac{1}{\omega^2} \diagdown \right] \{\ddot{q}\} \tag{7.32}$$

Considering a reduced number of normal coordinates, the solution by the mode-acceleration method, Eq. (7.32), represents an improvement over the solution by the mode-displacement method, Eq. (7.6), in that the response to the external forces has been made exact.

EXAMPLE 7.4

In Example 7.2 we obtained the exact solution for the forced vibration of the lumped-mass beam of Fig. 7-1, using the mode-displacement method. Assuming that we wish to obtain an approximate solution based on the fundamental mode only, let us reconsider this problem using the mode-acceleration method.

The displacements resulting from the external force F may be determined by use of Eq. (7.14). However, since we have not determined the stiffness coefficients, let us solve for the displacements directly. From elementary beam theory, we can obtain the displacements, leading to

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{ex} = \frac{Fl^3}{3EI}\begin{Bmatrix} \frac{4}{27} \\ \frac{14}{27} \\ 1 \end{Bmatrix}$$

Using the result given in Example 7.2 for the response in the fundamental normal coordinate q_1, we can write

$$\begin{aligned} q_{1,in} &= -\frac{\ddot{q}_1}{\omega_1^2} \\ &= -0.328\frac{Fl^3}{EI}\cos\omega_1 t \end{aligned}$$

From the second of Eqs. (7.18), the displacement response of the beam to the inertia forces in the fundamental mode is given by

$$\{x\}_{in} = \{\phi\}_1 q_{1,in}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{in} = -\frac{Fl^3}{EI}\begin{Bmatrix} 0.0531 \\ 0.177 \\ 0.328 \end{Bmatrix}\cos\omega_1 t$$

Then the total displacement response of the beam by the mode-acceleration method is given by

$$\{x\} = \{x\}_{ex} + \{x\}_{in}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{Fl^3}{EI}\begin{Bmatrix} 0.0494 - 0.0531\cos\omega_1 t \\ 0.173 - 0.177\cos\omega_1 t \\ 0.333 - 0.328\cos\omega_1 t \end{Bmatrix}$$

By comparison, the displacement response given by the mode-displacement method can be written as

$$\{x\} = \{\phi\}_1 q_1$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{Fl^3}{EI}\begin{Bmatrix} 0.0531 \\ 0.177 \\ 0.328 \end{Bmatrix}(1 - \cos\omega_1 t)$$

Note that the differences are small. Comparison of the solutions for the displacement x_3 of the end mass with the exact solution of Example 7.2 verifies that the mode acceleration method yields a more complete solution.

Suppose we are interested in the bending moment response at the root of the beam. By inspection of Fig. 7-1, the bending moment at the root resulting from the external force F is given simply by

$$L_{ex} = Fl$$

From Eq. (7.30), we can write the bending moment at the root resulting from the inertia forces in the fundamental mode as

$$\begin{aligned} L_{in} &= l_1 q_{1,in} \\ &= -1.12Fl \cos \omega_1 t \end{aligned}$$

Then, the mode-acceleration method gives the result

$$\begin{aligned} L &= L_{ex} + L_{in} \\ &= Fl(1 - 1.12Fl \cos \omega_1 t) \end{aligned}$$

By comparison, the result given by the mode displacement method of Example 7.2 is

$$\begin{aligned} L &= l_1 q_1 \\ &= 1.12Fl(1 - \cos \omega_1 t) \end{aligned}$$

While the differences between the solutions are not large, they are relatively much larger than the corresponding difference between the displacement solutions given earlier. The solution given by the mode-acceleration method is more complete as may be verified by comparison with the exact solution given in Example 7.2.

EXAMPLE 7.5

In Example 7.3 the rotations of the four disks of the system of Fig. 7-2 were represented approximately in terms of two normal coordinates. These coordinates were determined from an approximate representation of the motion. Based on the solution of the two uncoupled equations of forced vibration, the mode-displacement solutions were given for the rotation of the second disk and for the torque between the second and third disks. Let us reconsider this problem using the mode-acceleration method.

From inspection of Fig. 7-2, we can write that the rotation of the second disk resulting from the external torque T is given by

$$\theta_{2,ex} = \frac{T}{k}$$

From Eq. (7.23) and the solutions given in Example 7.3 for the response of the normal coordinates, we can write that

$$q_{1,in} = -1.339\frac{T}{k}\frac{\sin\omega_1 t}{\omega_1 t}$$

$$q_{2,in} = 0.335\frac{T}{k}\frac{\sin\omega_2 t}{\omega_2 t}$$

According to the second of Eqs. (7.18), the rotation of the second disk resulting from the inertia torques is given by

$$\begin{aligned}\theta_{2,in} &= \frac{T}{k}\lfloor 0.707 \;\; -0.467\rfloor\begin{Bmatrix}-1.339\dfrac{\sin\omega_1 t}{\omega_1 t}\\ 0.335\dfrac{\sin\omega_2 t}{\omega_2 t}\end{Bmatrix}\\ &= \frac{T}{k}\left(-0.946\frac{\sin\omega_1 t}{\omega_1 t} - 0.156\frac{\sin\omega_2 t}{\omega_2 t}\right)\end{aligned}$$

Then the total response of the second disk is

$$\theta_2 = \frac{T}{k}\left(1 - 0.946\frac{\sin\omega_1 t}{\omega_1 t} - 0.156\frac{\sin\omega_2 t}{\omega_2 t}\right)$$

By comparison, the result given by the mode displacement method in Example 7.3 is

$$\theta_2 = \frac{T}{k}\left(1.102 - 0.946\frac{\sin\omega_1 t}{\omega_1 t} - 0.156\frac{\sin\omega_2 t}{\omega_2 t}\right)$$

Referring to Fig. 7-2, it is evident that the elastic torque between the second and third disks resulting from the external torque is

$$L_{ex} = 0$$

From Eq. (7.30) and the results of Example 7.3, we can write the torque in the shaft between the second and third disks as

$$\begin{aligned}L_{in} &= \lfloor l_1 \; l_2\rfloor\begin{Bmatrix}q_1\\ q_2\end{Bmatrix}_{in}\\ &= T\lfloor 0.197 \; 0.491\rfloor\begin{Bmatrix}-1.339\dfrac{\sin\omega_1 t}{\omega_1 t}\\ 0.335\dfrac{\sin\omega_2 t}{\omega_2 t}\end{Bmatrix}\\ &= T\left(-0.264\frac{\sin\omega_1 t}{\omega_1 t} + 0.164\frac{\sin\omega_2 t}{\omega_2 t}\right)\end{aligned}$$

Since $L_{ex} = 0$, we can write that

$$L = L_{in}$$

The mode displacement solution reproduced from Example 7.3 is

$$L = T\left(-0.100 - 0.264\,\frac{\sin \omega_1 t}{\omega_1 t} + 0.164\,\frac{\sin \omega_2 t}{\omega_2 t}\right)$$

Thus the mode-displacement method gives the fairly large value of $-0.100T$ for the response to the external torque T whereas the exact response is zero. Evidently the mode-acceleration method produces a substantial improvement in the solution.

7.3 Systems with Rigid-Body Degrees of Freedom

If a system has rigid-body degrees of freedom, we can separate the displacement response into two parts, given by

$$\{x\} = \{x\}^R + \{x\}^E \tag{7.33}$$

In which the superscripts R and E indicate the rigid-body and elastic displacements. Rigid-body motion is described by a number of the normal coordinates equal to the number of rigid-body degrees of freedom. The remaining normal coordinates describe the elastic motion. In the mode-displacement method, the displacement response is given by

$$\begin{aligned} \{x\}^R &= [\phi]^R\{q\}^R \\ \{x\}^E &= [\phi]^E\{q\}^E \end{aligned} \tag{7.34}$$

or by

$$\{x\} = \left[[\phi]^R \,\vdots\, [\phi]^E\right]\left\{\begin{matrix}\{q\}^R \\ \hdashline \{q\}^E\end{matrix}\right\} \tag{7.35}$$

From physical reasoning, it is evident that the internal elastic forces in a rigid-body mode will be zero. We can write that $\lfloor l \rfloor^R = \lfloor 0 \rfloor$. Referring to Eq. (7.6), we can write the internal elastic force L at a selected point of the system as

$$L = \lfloor l \rfloor^E\{q\}^E \tag{7.36}$$

From comparison of Eqs. (7.35) and (7.36) with Eqs. (7.2) and (7.6), it is apparent that the presence of rigid-body modes requires no changes in the mode displacement method.

If we are considering a reduced number of coordinates, we may want to use the mode-acceleration method. Let us assume that we have obtained the response in all of the rigid-body modes and can use the first of Eqs. (7.34) to give the rigid-body motion. We can separate the elastic displacement response into two parts as given by

$$\{x\}^E = \{x\}^E_{ex+in_R} + \{x\}^E_{in_E} \tag{7.37}$$

in which the first term on the right-hand side represents the elastic displacements resulting from the external forces and the rigid-body inertia forces.

The second term on the right-hand side of Eq. (7.37) represents the elastic displacements resulting from the inertia forces due to the elastic motion. Referring to the equations of motion, Eq. (7.1), we can define the response to the external forces and the rigid-body inertia forces by

$$-[m]\{\ddot{x}\}^R - [k]\{x\}^E_{ex+in_R} + \{F\}_{ex} = \{0\} \tag{7.38}$$

Since the stiffness matrix is singular for a system with rigid-body degrees of freedom, we are unable to solve this equation directly for the displacements. However, the elastic displacements must be orthogonal to the rigid-body displacements, as represented by

$$[\phi]^{TR}[m]\{x\}^E = \{0\} \tag{7.39}$$

There are as many of these equations as there are rigid-body degrees of freedom. If we use Eq. (7.39) to reduce the number of equations in Eq. (7.38), we can obtain a solution From Eq (7.24), we can write the elastic-displacement response to the inertia forces resulting from elastic motion as

$$\{x\}^E_{in_E} = -[\phi]^E \left[\diagdown \frac{1}{\omega^2} \diagdown\right]^E \{\ddot{q}\}^E \tag{7.40}$$

The elastic response, determined from Eqs. (7.37) through (7.40), is more complete than the result given by the mode-displacement method, the second of Eqs. (7.34). As before, the difference in the methods lies in the completeness with which the response to the external forces is included

Let us consider that the internal elastic force L at a point of the system consists of two parts, as given by

$$L = L_{ex+in_R} + L_{in_E} \tag{7.41}$$

We can determine the elastic force L_{ex+in_R} by applying the external and rigid-body inertia forces and using the methods of mechanics of materials From Eqs. (7.31), the elastic force L_{in_E} is given by

$$L_{in_E} = -[l]^E \left[\diagdown \frac{1}{\omega^2} \diagdown\right]^E \{\ddot{q}\} \tag{7.42}$$

Then we can write

$$L = L_{ex+in_R} - [l]^E \left[\diagdown \frac{1}{\omega^2} \diagdown\right]^E \{\ddot{q}\}^E \tag{7.43}$$

EXAMPLE 7.6

An airplane wing is idealized as a massless uniform beam with attached lumped masses as shown in Fig. 7-3 The airplane experiences a landing impact in which the force is assumed to be a half-period sine pulse as shown. Since the loading is symmetric, a symmetric response will result and we need

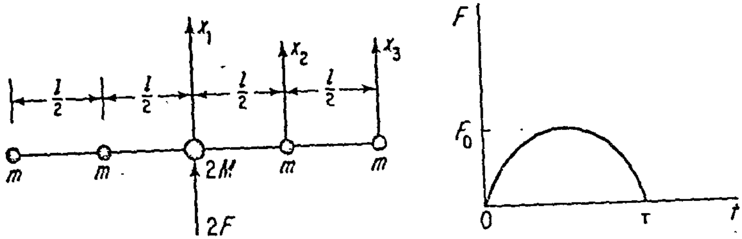


Fig. 7-3 *Idealized airplane excited by a landing impact.*

only consider half of the system. For the case in which $m/M = 0.1$, the first two natural frequencies and normal mode shapes are given by

$$\omega_1^2 = 0$$

$$\omega_2^2 = 3.14 \frac{EI}{ml^3}$$

$$[\phi] = \begin{bmatrix} 1 & -0.123 \\ 1 & 0.234 \\ 1 & 1 \end{bmatrix}$$

The first two generalized masses are

$$[M] = \begin{bmatrix} 1 & 1 & 1 \\ -0.123 & 0.234 & 1 \end{bmatrix} \begin{bmatrix} 10m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} 1 & -0.123 \\ 1 & 0.234 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 12m & 0 \\ 0 & 1.209m \end{bmatrix}$$

Then the generalized stiffnesses can be written as

$$K_1 = M_1\omega_1^2 = 0$$

$$K_2 = M_2\omega_2^2 = 1.209m\omega_2^2$$

The generalized external forces are given by

$$F_{1,ex} = F$$

$$F_{2,ex} = -0.123F$$

We can write the equations of motion for the first two normal coordinates as

$$12m\ddot{q}_1 = F$$

$$1.209m\ddot{q}_2 + 1.209m\omega_2^2 q_2 = -0.123F$$

If we are interested in the bending moment at the root of the wing, we will only need to obtain the solution for the second normal coordinate. Suppose that the duration of the pulse is equal to the period of the first elastic period of the system, as given by $\omega_2\tau = 2\pi$. Making use of the results of Example

3.20, we can write, for $0 < t < \tau$, that

$$q_2 = \frac{\dfrac{-0.123F_0}{1.209m\omega_2^2}}{1 - \dfrac{\pi^2}{4\pi^2}}\left(\sin\frac{\pi t}{\tau} - \frac{\pi}{2\pi}\sin\omega_2 t\right)$$

$$= -0.137\frac{F_0}{m\omega_2^2}\left(\sin\frac{\pi t}{\tau} - \frac{1}{2}\sin\omega_2 t\right)$$

and, for $\tau < t$, that

$$q_2 = -\frac{\dfrac{-0.123F_0}{1.209m\omega_2^2}}{1 - \dfrac{\pi^2}{4\pi^2}}\sin\omega_2 t$$

$$= 0.137\frac{F_0}{m\omega_2^2}\sin\omega_2 t$$

Let us determine the bending moment at the root per unit displacement of the normal coordinates. Since the first mode is a rigid-body mode, there can be no internal elastic forces and we can write $l_1 = 0$. For the second mode, we can obtain the bending moment from the inertia loading. We can write

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \omega_2^2\begin{bmatrix} 10m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{Bmatrix} -0.123 \\ 0.234 \\ 1 \end{Bmatrix}$$

$$= m\omega_2^2\begin{Bmatrix} -1.23 \\ 0.234 \\ 1 \end{Bmatrix}$$

The resulting inertia loading is shown on Fig. 7-4. Then the bending moment at the root per unit displacement of the second normal coordinate is given by

$$l_2 = \tfrac{1}{2}l \times 0.234m\omega_2^2 + lm\omega_2^2$$
$$= 1.117ml\omega_2^2$$
$$= 3.51\frac{EI}{l^2}$$

The bending moment at the root can be written as

$$L = \frac{EI}{l^3}\lfloor 0 \;\; 3.51 \rfloor\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

or, making use of the solution for q_2, as

$$L = -0.153F_0 l\left(\sin\frac{\pi t}{\tau} - \frac{1}{2}\sin\omega_2 t\right) \qquad \text{for } 0 < t < \tau$$
$$= 0.153F_0 l\sin\omega_2 t \qquad \text{for } \tau < t$$

This is the solution given by the mode-displacement method.

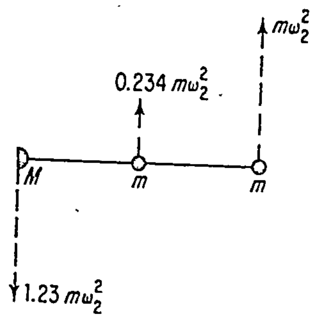


Fig. 7-4 *Inertia loading per unit displacement of the second normal coordinate.*

In applying the mode-acceleration method, we must obtain the bending moment at the root resulting from the external force and the rigid-body inertia forces shown on Fig. 7-5. The bending moment at the root is given by

$$\begin{aligned} L_{ex+in_R} &= -\frac{l}{2}\cdot\frac{F}{12} - l\frac{F}{12} = -\frac{1}{8}Fl \\ &= -\frac{1}{8}F_0 l \sin\frac{\pi t}{\tau} \qquad \text{for } 0 < t < \tau \\ &= 0 \qquad \text{for } \tau < t \end{aligned}$$

From Eq. (7.23), the response of the second normal coordinate to the inertia forces is given by

$$\begin{aligned} q_{2,in} &= -\frac{\ddot{q}_2}{\omega_2^2} \\ &= -0.0343\frac{F_0}{m\omega_2^2}\left(\sin\frac{\pi t}{\tau} - 2\sin\omega_2 t\right) \qquad \text{for } 0 < t < \tau \\ &= 0.137\frac{F_0}{m\omega_2^2}\sin\omega_2 t \qquad \text{for } \tau < t \end{aligned}$$

For the bending moment at the root of the wing, the mode-acceleration method gives, for $0 < t < \tau$, the result

$$\begin{aligned} L &= L_{ex+in_R} + l_2 q_{2,in} \\ &= -\tfrac{1}{8}F_0 l \sin\frac{\pi t}{\tau} - 0.0383F_0 l\left(\sin\frac{\pi t}{\tau} - 2\sin\omega_2 t\right) \\ &= -0.163F_0 l \sin\frac{\pi t}{\tau} + 0.0766F_0 l\sin\omega_2 t \end{aligned}$$

and, for $\tau < t$, the result

$$L = 0.153F_0 l\sin\omega_2 t$$

In the period of loading, $0 < t < \tau$, the results of the two methods differ in the term involving $\sin\frac{\pi t}{\tau}$. This term represents the response to the external

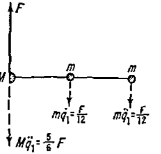


Fig. 7-5 *External forces and rigid-body inertia forces.*

force. In the mode-acceleration method, this term is exact, including the, contribution of the third mode. After the loading period, described by $\tau < t$, the two methods yield the same result

7 4 Displacement Excitation

Sometimes the excitation may be given as a prescribed motion rather than as a force. Suppose that the forced vibrations are excited by a given motion of the nth coordinate, x_n. We can write the equations of motion as

$$\{\sum F\} = -[m]\{\ddot{x}\} - [k]\{x\} + \{F\}_{ex} = \{0\} \tag{7.44}$$

in which

$$\{F\}_{ex} = \begin{Bmatrix} 0 \\ \\ 0 \\ \hline F_n \end{Bmatrix}_{ex} \tag{7.45}$$

The force $F_{n,ex}$ which produces the known motion of x_n is an unknown of the problem However, the system now has $n - 1$ degrees of freedom and we can drop the nth equation of motion. The remaining equations are

$$\begin{aligned} \begin{Bmatrix} \sum F_1 \\ \cdot \\ \sum F_{n-1} \end{Bmatrix} = &-\begin{bmatrix} m_{11} & & m_{1,n-1} \\ & & \\ m_{n-1,1} & & m_{n-1\ n-1} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \\ \ddot{x}_{n-1} \end{Bmatrix} \\ &- \begin{bmatrix} k_{11} & & k_{1,n-1} \\ & & \\ k_{n-1,1} & & k_{n-1\ n-1} \end{bmatrix} \begin{Bmatrix} x_1 \\ \\ x_{n-1} \end{Bmatrix} - \begin{Bmatrix} m_{1n} \\ \\ m_{n-1,n} \end{Bmatrix} \ddot{x}_n \\ &- \begin{Bmatrix} k_{1n} \\ \\ k_{n-1,n} \end{Bmatrix} x_n = \begin{Bmatrix} 0 \\ \\ 0 \end{Bmatrix} \end{aligned} \tag{7 46}$$

If we regard the terms involving the prescribed motion x_n as a set of external forces, we can write that

$$\begin{Bmatrix} F_1 \\ \vdots \\ F_{n-1} \end{Bmatrix}_{ex} = -\begin{Bmatrix} m_{1n} \\ \vdots \\ m_{n-1,n} \end{Bmatrix}\ddot{x}_n - \begin{Bmatrix} k_{1n} \\ \vdots \\ k_{n-1,n} \end{Bmatrix}x_n \tag{7.47}$$

Thus, Eq. (7.46) represents a set of $n - 1$ equations of motion in the $n - 1$ coordinates $x_1, x_2, \ldots, x_{n-1}$, in which the exciting forces are known. We can obtain a solution using our usual methods. The normal mode shapes needed to define the normal coordinates can be obtained from the solution of the homogeneous part of Eq. (7.46). Note that this set of equations represents the free vibrations of the system with a constraint added which prohibits motion in x_n.

Suppose that one of the constraints of a system is given a prescribed motion. Let us introduce a new coordinate x_{n+1}, which describes the motion of the constraint. We can proceed to write a set of $n + 1$ equations of motion having the form of Eq. (7.44). Following the procedure just outlined, we can drop the $(n + 1)$th equation of motion, leading to a set of n equations having the form of Eq. (7.46). Knowing the exciting forces, we can solve for the motion. For this case the normal mode shapes needed are those for the system with the constraint fixed, represented by $x_{n+1} = 0$.

We can extend the methods just outlined to account for the prescribed motion of more than one degree of freedom or constraint. For the special case in which the constraint system moves as a rigid unit, we would most likely describe the motion of the system relative to the constraint system. Then the external forces acting on the system would be the inertia forces resulting from the constraint motion.

EXAMPLE 7.7

The fourth disk of the torsional system of Fig. 6-4 is given a harmonic motion $\theta_4 = \theta_0 \sin \omega t$ as shown in Fig. 7-6. The equations of motion are

$$I\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \end{Bmatrix} + k\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ T_4 \end{Bmatrix}_{ex}$$

in which the external torque $T_{4,ex}$ is not known. Dropping the fourth equation, we can write

$$I\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = k\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}\theta_4$$

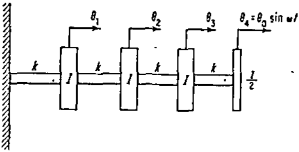


Fig. 7-6

The homogeneous part of this set of equations represents the equations for the free vibrations of the system with the fourth disk fixed, as in Fig. 7-7. The natural frequencies and normal mode shapes for this system are

$$\omega_1^2 = (2 - \sqrt{2})\frac{k}{I}$$
$$\omega_2^2 = 2\frac{k}{I}$$
$$\omega_3^2 = (2 + \sqrt{2})\frac{k}{I}$$
$$[\phi] = \begin{bmatrix} 1 & -1 & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & 1 & 1 \end{bmatrix}$$

Transformation of the equations of motion to the normal coordinates results in

$$4I\ddot{q}_1 + (8 - 4\sqrt{2})kq_1 = k\theta_0 \sin \omega t$$
$$2I\ddot{q}_2 + 4kq_2 = k\theta_0 \sin \omega t$$
$$4I\ddot{q}_3 + (8 + 4\sqrt{2})kq_3 = k\theta_0 \sin \omega t$$

We can write the solution for the steady-state motion of the system as

$$q_1 = \frac{k}{(8 - 4\sqrt{2})k - 4I\omega^2}\,\theta_0 \sin \omega t = \frac{\frac{1}{4}\left(1 + \frac{\sqrt{2}}{2}\right)}{1 - \frac{\omega^2}{\omega_1^2}}\,\theta_0 \sin \omega t$$

$$q_2 = \frac{k}{4k - 2I\omega^2}\,\theta_0 \sin \omega t = \frac{\frac{1}{4}}{1 - \frac{\omega^2}{\omega_2^2}}\,\theta_0 \sin \omega t$$

$$q_3 = \frac{k}{(8 + 4\sqrt{2})k - 4I\omega^2}\,\theta_0 \sin \omega t = \frac{\frac{1}{4}\left(1 - \frac{\sqrt{2}}{2}\right)}{1 - \frac{\omega^2}{\omega_3^2}}\,\theta_0 \sin \omega t$$

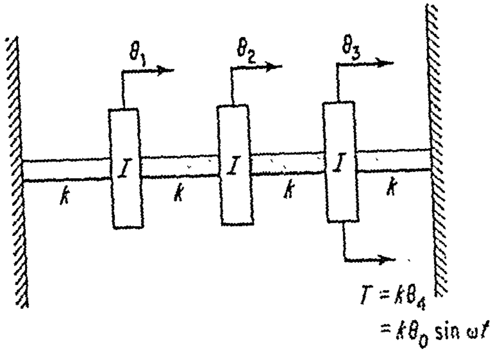


Fig. 7-7 *Forced-vibration problem equivalent to that of Fig. 7-6.*

The solution for the rotation of the third disk, taken as an example, is given by

$$\begin{aligned}\theta_3 &= \lfloor 1 \;\; 1 \;\; 1 \rfloor \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} \\ &= \left[\frac{1 + \frac{\sqrt{2}}{2}}{1 - \frac{\omega^2}{\omega_1^2}} + \frac{1}{1 - \frac{\omega^2}{\omega_2^2}} + \frac{1 - \frac{\sqrt{2}}{2}}{1 - \frac{\omega^2}{\omega_3^2}} \right] \frac{\theta_0}{4} \sin \omega t\end{aligned}$$

For comparison, let us apply the mode-acceleration method. Referring to Fig. 7-7, the rotation of the third disk resulting from the external torque is given by

$$\theta_{3,ex} = \tfrac{3}{4}\theta_0 \sin \omega t$$

We can write the response of the fundamental normal coordinate to the inertia forces as

$$\begin{aligned}q_{1,in} &= -\frac{\ddot{q}_1}{\omega_1^2} \\ &= \frac{1}{4}\left(1 + \frac{\sqrt{2}}{2}\right) \frac{\frac{\omega^2}{\omega_1^2}}{1 - \frac{\omega^2}{\omega_1^2}} \theta_0 \sin \omega t\end{aligned}$$

Then the result given by the mode-acceleration method using the fundamental mode only is

$$\begin{aligned}\theta_3 &= \theta_{3,ex} + \phi_{31} q_{1,in} \\ &= \left[\frac{3}{4} + \frac{1}{4}\left(1 + \frac{\sqrt{2}}{2}\right) \frac{\frac{\omega_2}{\omega_1^2}}{1 - \frac{\omega^2}{\omega_1^2}} \right] \theta_0 \sin \omega t\end{aligned}$$

For comparison, the result given by the mode-displacement method with the fundamental mode only is

$$\theta_3 = \frac{\frac{1}{4}(1+\sqrt{2})}{1-\dfrac{\omega^2}{\omega_1^2}}\theta_0 \sin \omega t$$

$$= \frac{1}{4}\left(1+\frac{\sqrt{2}}{2}\right)\left[1+\frac{\dfrac{\omega^2}{\omega_1^2}}{1-\dfrac{\omega^2}{\omega_1^2}}\right]\theta_0 \sin \omega t$$

EXAMPLE 7.8

The third mass of a triple pendulum is given a transient horizontal motion $x_3(t)$ of small amplitude as shown in Fig. 7-8. In Prob 5-5, the motion of the system was described in terms of the angles θ_1, θ_2, θ_3. Here it will be more convenient to use the displacements x_1, x_2, x_3. The equations of motion for the system are given by

$$m\begin{Bmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \ddot{x}_3\end{Bmatrix} + \frac{mg}{l}\begin{bmatrix}5 & -2 & 0\\ -2 & 3 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix} = \begin{Bmatrix}0\\ 0\\ F_3\end{Bmatrix}_{ex}$$

The external force $F_{3,ex}$ is unknown. If we drop the third equation, we can write

$$m\begin{Bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{Bmatrix} + \frac{mg}{l}\begin{bmatrix}5 & -2\\ -2 & 3\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix} = \frac{mg}{l}\begin{Bmatrix}0\\ 1\end{Bmatrix}x_3$$

The homogeneous part of the equations of motion represents the free vibrations of the system with the constraint $x_3 = 0$ as in Fig. 7-9. Determining the natural frequencies and normal mode shapes, we can write

$$\omega_1^2 = (4-\sqrt{5})\frac{g}{l}$$

$$\omega_2^2 = (4+\sqrt{5})\frac{g}{l}$$

$$[\phi] = \begin{bmatrix}\frac{1}{2}(\sqrt{5}-1) & -\frac{1}{2}(\sqrt{5}+1)\\ 1 & 1\end{bmatrix}$$

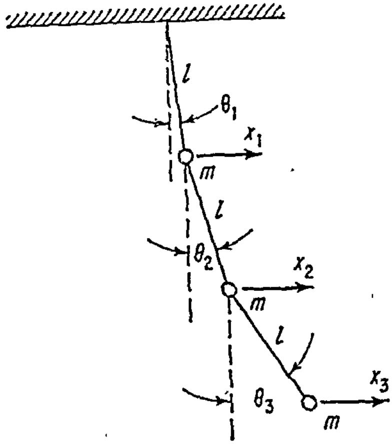


Fig. 7-8

Transforming to the normal coordinates, the equations of motion become

$$\tfrac{1}{2}(5 - \sqrt{5})m\ddot{q}_1 + \tfrac{1}{2}(25 - 9\sqrt{5})\frac{mg}{l}q_1 = \frac{mg}{l}x_3$$

$$\tfrac{1}{2}(5 + \sqrt{5})m\ddot{q}_2 + \tfrac{1}{2}(25 + 9\sqrt{5})\frac{mg}{l}q_2 = \frac{mg}{l}x_3$$

For any given motion $x_3(t)$ we can obtain the response of the normal coordinates and the motion of the upper two masses.

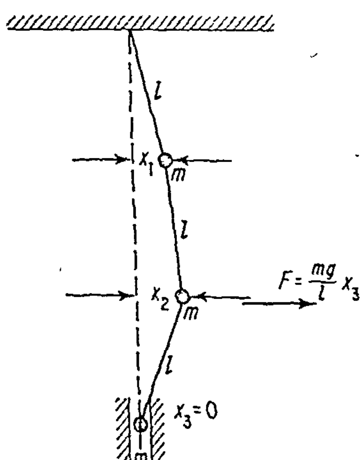
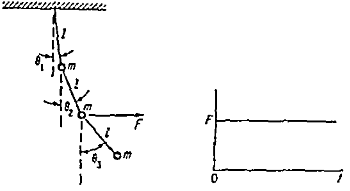


Fig. 7-9 *Forced-vibration problem equivalent to that of Fig. 7-8.*

Problems

7-1 A step force is applied to the second mass of the triple pendulum of Prob. 5-5 as shown. Determine the displacement response of the system, using the mode-displacement method.

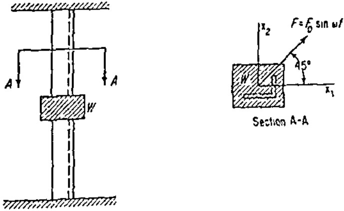


Prob. 7-1

7-2 A bar having an unequal angle cross section is clamped at its ends as shown. The mass of the bar is small compared with the mass of the attached weight $W = 10$ lb. The flexibility influence coefficients corresponding to the displacements x_1, x_2 of the weight are given by

$$[C] = \begin{bmatrix} 0.02 & 0.03 \\ 0.03 & 0.10 \end{bmatrix} \text{in/.lb}$$

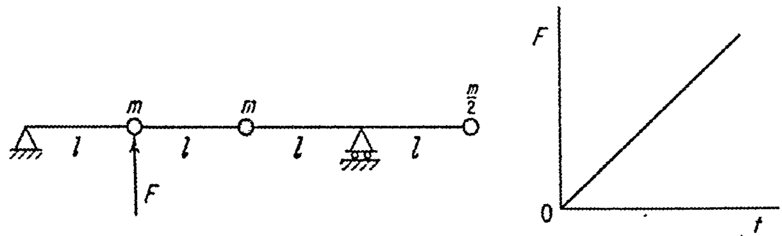
A horizontal force $F = F_0 \sin \omega t$ with $F_0 = 100$ lb and $\omega = 15$ rad/sec is applied to the weight as shown. Determine the steady-state displacement



Prob. 7-2

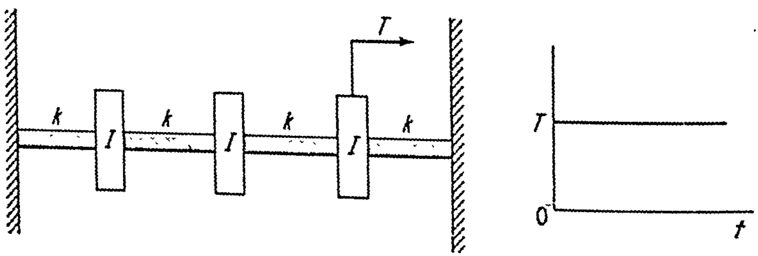
response in the coordinates x_1,x_2. Obtain the shear force transmitted to the angle, using the mode-displacement method.

7-3 A ramp force F is applied to the first mass of the lumped mass beam of Prob. 6-1 as shown. Using the mode-displacement method, determine the bending moment in the fundamental mode at the right hand support.



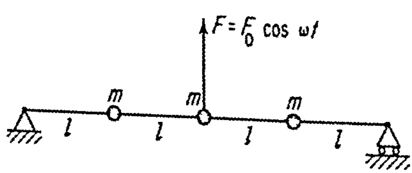
Prob. 7-3

7-4 A step torque is applied to the third disk of the torsional system shown. Determine the displacement response of the third disk. Use the mode-displacement method to obtain the torque in the first shaft segment.



Prob. 7-4

7-5 A harmonically varying force is applied to the midpoint of the lumped mass beam of Prob. 5-9 as shown. Determine the shear force in the fundamental mode at the left-hand support, using the mode-displacement method.



Prob. 7-5

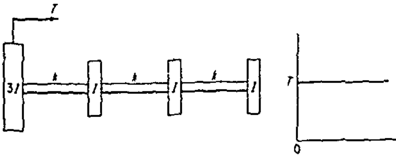
7-6 Using the mode-acceleration method, determine the shear force transmitted to the angle of Prob. 7-2. Consider only the response in the fundamental mode. Compare the result with that given by the fundamental mode, using the mode-displacement method as in Prob. 7-2.

7-7 For the system of Prob. 7-3, obtain the bending moment at the right-hand support, using the response in the fundamental mode and the mode-acceleration method.

7-8 Determine the displacement response of the third disk of the torsional system of Prob. 7-4, using the mode-acceleration method. Consider only the response in the fundamental mode Compare the result with that given by the mode-displacement method using only the fundamental mode.

7-9 Using the mode-acceleration method, obtain the shear force at the left-hand support of the beam of Prob. 7-5. As before, use only the response in the fundamental mode. Compare the result with that of Prob. 7-5.

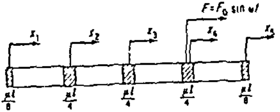
7-10 A step torque is applied to the first disk of the torsional system of Prob. 5-2 as shown. Determine the torque in the last shaft segment, using the mode-displacement method.



Prob. 7-10

7-11 For the system of Prob 7-10, obtain the torque in the last shaft segment, using the mode-acceleration method. Consider only the response in the first elastic mode.

7-12 An axial harmonic force is applied to the fourth lumped mass of the longitudinal system of Prob 6-7 as shown Using the mode-displacement method and the steady-state response in the first two modes, determine the displacement x_4 and the tension in the fourth segment



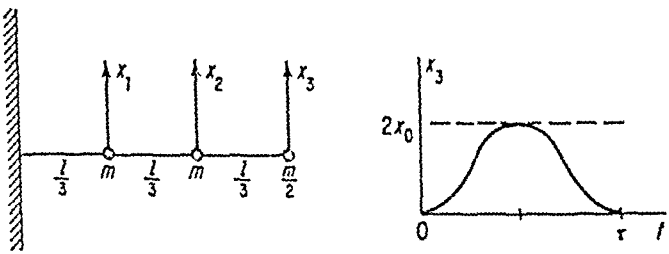
Prob. 7-12

7-13 Determine the tension in the fourth segment of the longitudinal system of Prob. 7-12, using the mode-acceleration method. Consider the response in the first two modes. Compare the result with that for the mode-displacement method found in Problem 7-12.

7-14 The lumped-mass beam of Example 7.2 is excited as shown by the prescribed motion of the third mass given by

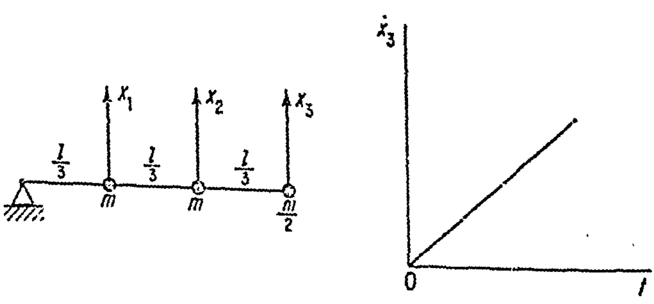
$$\begin{aligned} x_3 &= x_0\left(1 - \cos\frac{2\pi t}{\tau}\right) & &\text{for } 0 < t < \tau \\ &= 0 & &\text{for } \tau < t \end{aligned}$$

Determine the motion of the first two masses, described by x_1, x_2. Obtain the shear force in the third beam section.



Prob. 7-14

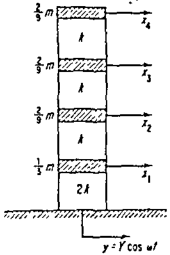
7-15 A rocket is idealized as a lumped-mass beam of flexural rigidity EI as shown. The rocket is being erected by a hoist at the right hand end which raises the end with a linearly increasing velocity $\dot{x}_3$ as shown. Determine the maximum bending moment in the rocket.



Prob. 7-15

7-16 The idealized building of Prob. 6-10 is excited by the horizontal ground motion $y = Y\cos\omega t$ as shown. Find the steady-state shear force at

the base considering only the response of the fundamental mode. Use both the mode-displacement and mode-acceleration methods.



Prob. 7-16

CHAPTER 8

Free Vibrations of a Continuous System

8.1 Differential Equations of Motion

(a) Longitudinal or Torsional Vibrations of a Rod

Let us consider the longitudinal vibrations of a slender straight elastic rod. If we assume that all of the particles at a cross section have equal longitudinal displacement, we can describe the longitudinal motion of the bar by $u(x,t)$ as shown in Fig. 8-1. An element of the rod of infinitesimal length dx is shown. The cross sections originally at x and $x + dx$ are displaced by u and by $u + \frac{\partial u}{\partial x} dx$. The forces acting on the ends of the element by the adjoining elements are shown on the free-body diagram as F and $F + \frac{\partial F}{\partial x} dx$. Letting f_{ex} represent the external force acting per unit length of the rod, we can write the axial external force on the element as $f_{ex}\, dx$. Representing the mass per unit length of the rod by μ, the mass of the element is $\mu\, dx$. In general, the rod may have a variable cross section and μ may be variable with x.

Referring to the free-body diagram, we can write the equation for the longitudinal motion of the element as

$$\mu\, dx \frac{\partial^2 u}{\partial t^2} = F + \frac{\partial F}{\partial x} dx - F + f_{ex}\, dx$$

or, simplifying, as

$$\frac{\partial F}{\partial x} - \mu \frac{\partial^2 u}{\partial t^2} + f_{ex} = 0 \tag{8.1}$$

We will assume that the axial strain $\frac{\partial u}{\partial x}$ is small and that the material is elastic. Then we can write the stress-strain relationship as

$$\frac{F}{A} = E \frac{\partial u}{\partial x} \tag{8.2}$$

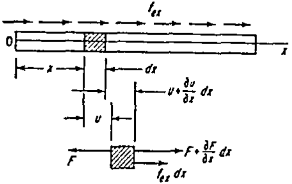


Fig. 8-1 *Element of rod in longitudinal vibration.*

in which A and E are the cross-sectional area and the modulus of elasticity. From Eqs. (8.1) and (8.2), we can write

$$\frac{\partial}{\partial x}\left(EA\,\frac{\partial u}{\partial x}\right) - \mu\,\frac{\partial^2 u}{\partial t^2} + f_{ex} = 0 \tag{8.3}$$

This is the partial differential equation for the longitudinal vibration of the rod. It will be convenient to represent $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t}$ by a prime and a dot respectively. Then we can write the equation of motion as

$$(EAu')' - \mu\ddot{u} + f_{ex} = 0 \tag{8.4}$$

For a rod of length l, the conditions at the boundaries $x = 0$ and $x = l$ must be given. We can specify either the displacement or the force at the ends, as given by

$$\begin{array}{lcl} u(0, t) & \text{or} & F_{ex}(0, t) \\ u(l, t) & \text{or} & F_{ex}(l, t) \end{array} \tag{8.5}$$

For a free vibration, $f_{ex} = 0$, and the displacement or force at the ends will be zero. For this case, we can write

$$\begin{array}{lcl} u(0, t) = 0 & \text{or} & F_{ex}(0, t) = EAu'(x, t)|_{x=0} = 0 \\ u(l, t) = 0 & \text{or} & F_{ex}(l, t) = EAu'(x,t)|_{x=l} = 0 \end{array} \tag{8.6}$$

The torsional vibrations of the elastic rod being considered can be described by the rotation $\theta(x,t)$ around the axis as shown in Fig. 8-2. Consider the torques acting on the element of infinitesimal length dx. The torques exerted by the neighboring elements are given as T and $T + \frac{\partial T}{\partial x}\,dx$. Letting τ_{ex} represent the external torque acting per unit length of the rod, we can write the external torque in the element as $\tau_{ex}\,dx$. If the axial moment of

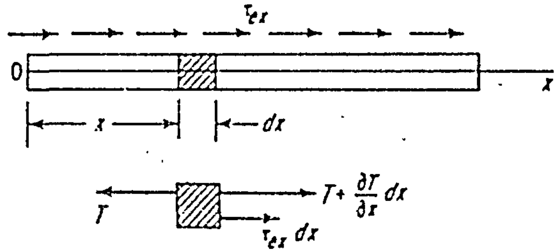


Fig. 8-2 *Element of rod in torsional vibration.*

inertia per unit length of the rod is given by I, the axial moment of inertia of the element is $I\,dx$.

From the free-body diagram, we can write the equation for the angular motion of the element as

$$I\,dx\,\frac{\partial^2\theta}{\partial t^2} = T + \frac{\partial T}{\partial x}\,dx - T + \tau_{ex}\,dx$$

which leads to

$$\frac{\partial T}{\partial x} - I\frac{\partial^2\theta}{\partial t^2} + \tau_{ex} = 0 \tag{8.7}$$

If the shear strains are not large and the material is elastic, the torque-rotation relationship is

$$T = GJ\frac{\partial\theta}{\partial x} \tag{8.8}$$

in which GJ is the torsional rigidity of the rod. Substitution of Eq. (8.8) into Eq. (8.7) leads to

$$\frac{\partial}{\partial x}\left(GJ\frac{\partial\theta}{\partial x}\right) - I\frac{\partial^2\theta}{\partial t^2} + \tau_{ex} = 0 \tag{8.9}$$

Representing $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t}$ as a prime and a dot respectively, we can write the equation for torsional motion as

$$(GJ\theta')' - I\ddot{\theta} + \tau_{ex} = 0 \tag{8.10}$$

At the ends of the rod, given by $x = 0$ and $x = l$, we can specify either the rotation or the torque, as given by

$$\begin{aligned} &\theta(0, t) \quad \text{or} \quad T_{ex}(0, t) \\ &\theta(l, t) \quad \text{or} \quad T_{ex}(l, t) \end{aligned} \tag{8.11}$$

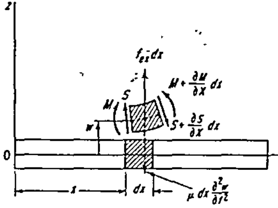


Fig. 8-3 *Element of beam in flexural vibration.*

For a free torsional vibration, $\tau_{ex} = 0$, and the rotation or torque at the ends will be zero, as given by

$$\begin{aligned} \theta(0,t) &= 0 \quad \text{or} \quad T_{ex}(0,t) = GJ\theta'(x,t)|_{x=0} = 0 \\ \theta(l,t) &= 0 \quad \text{or} \quad T_{ex}(l,t) = GJ\theta'(x,t)|_{x=l} = 0 \end{aligned} \tag{8.12}$$

Comparison of the equations of motion for longitudinal or torsional vibration, given by Eqs. (8.4) and (8.10), shows that the problems are mathematically identical. As a further example, the equation for the transverse motion of a stretched string has the same form as Eqs. (8.4) and (8.10).

(*b*) *Flexural Vibrations of a Beam*

Consider the flexural vibrations of a slender straight elastic beam such as is shown in Fig. 8-3. In the reference position, the elastic axis of the beam lies along the x axis with the ends at $x = 0$ and $x = l$. Neglecting shear deformation, the displacement of the beam is described by the displacement of the elastic axis, shown as $w(x,t)$. Let us examine the forces and moments acting on an element of infinitesimal length dx. The shear force exerted on the element by the adjacent elements are shown as S and $S + \frac{\partial S}{\partial x}dx$. Similarly, the bending moments acting are given by M and $M + \frac{\partial M}{\partial x}dx$. The external force acting on the element is $f_{ex}\,dx$ in which f_{ex} represents the transverse force per unit length. Letting μ represent the mass of the beam per unit length, the mass of the element is $\mu\,dx$.

Referring to the free-body diagram, we can write the equation for the transverse motion of the element as

$$\mu\,dx\,\frac{\partial^2 w}{\partial t^2} = -S - \frac{\partial S}{\partial x}dx + S + f_{ex}\,dx$$

or, simplifying, as

$$-\frac{\partial S}{\partial x} - \mu \frac{\partial^2 w}{\partial t^2} + f_{ex} = 0 \tag{8.13}$$

If we neglect the inertia moment resulting from rotation of the element, the moments acting must be in equilibrium, leading to

$$0 = M + \frac{\partial M}{\partial x} dx - M - S\, dx$$

Simplifying

$$\frac{\partial M}{\partial x} = S \tag{8.14}$$

In elementary beam theory, the bending moment is proportional to the curvature. If the slope $\frac{\partial w}{\partial x}$ remains small during the motion

$$M = EI \frac{\partial^2 w}{\partial x^2} \tag{8.15}$$

in which EI represents the bending rigidity of the beam and $\frac{\partial^2 w}{\partial x^2}$ approximates the curvature. Combining Eqs. (8.13) through (8.15), we can write the equation of motion for flexural vibrations as

$$-\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 w}{\partial x^2}\right) - \mu \frac{\partial^2 w}{\partial t^2} + f_{ex} = 0 \tag{8.16}$$

or as

$$-(EIw'')'' - \mu \ddot{w} + f_{ex} = 0 \tag{8.17}$$

At each end we can specify either the shear force or displacement and either the bending moment or slope. The boundary conditions required are summarized by

$$\begin{aligned} w(0,t) &\quad \text{or} \quad S_{ex}(0,t) \\ w'(0,t) &\quad \text{or} \quad M_{ex}(0,t) \\ w(l,t) = 0 &\quad \text{or} \quad S_{ex}(l,t) \\ w'(l,t) = 0 &\quad \text{or} \quad M_{ex}(l,t) \end{aligned} \tag{8.18}$$

For a free flexural vibration, $f_{ex} = 0$, and the specified quantities at the ends are zero, as indicated by

$$\begin{aligned} w(0,t) = 0 &\quad \text{or} \quad S_{ex}(0,t) = (EIw'')'|_{x=0} = 0 \\ w'(x,t)|_{x=0} = 0 &\quad \text{or} \quad M_{ex}(0,t) = EIw''(x,t)|_{x=0} = 0 \\ w(l,t) = 0 &\quad \text{or} \quad S_{ex}(l,t) = (EIw''(x,t))'|_{x=l} = 0 \\ w'(x,t)|_{x=l} = 0 &\quad \text{or} \quad M_{ex}(l,t) = EIw''(x,t)|_{x=l} = 0 \end{aligned} \tag{8.19}$$

Evidently the equation of motion for flexural vibrations has a different form than that for the longitudinal or torsional vibrations.

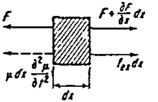


Fig. 8-4 *Longitudinal forces acting on an infinitesimal element of a rod.*

8.2 Derivation of the Equations of Motion Using the Energy Approach

To illustrate the approach, let us consider the longitudinal vibrations of the elastic rod of Fig. 8-1. Making use of D'Alembert's principle, the forces including the inertia force acting on an element of infinitesimal length dx are shown in Fig. 8-4 Consider a virtual displacement δu of the elements of the rod, illustrated by Fig. 8-5. The virtual displacements are required to be consistent with the displacement constraints. Suppose the displacement at $x = 0$ is specified in the problem as $u(0,t)$ Then, as shown, the virtual displacement $\delta u(0,t)$ must be zero. The virtual displacements are otherwise arbitrary except that it is usual to require them to be infinitesimal in magnitude.

We can write the strain energy in the rod before the virtual displacement as

$$U = \frac{1}{2}\int_0^l EA\left(\frac{\partial u}{\partial x}\right)^2 dx \tag{8.20}$$

in which E and A are the modulus of elasticity and the cross-sectional area. After the virtual displacement

$$U + \delta U = \frac{1}{2}\int_0^l EA\left[\frac{\partial(u + \delta u)}{\partial x}\right]^2 dx$$

in which δU represents the change in strain energy Considering $\frac{\partial(\delta u)}{\partial x}$ to be much smaller than $\frac{\partial u}{\partial x}$, we can reduce this expression to

$$U + \delta U = \frac{1}{2}\int_0^l EA\left[\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial u}{\partial x}\frac{\partial(\delta u)}{\partial x}\right] dx$$

Comparing this equation with Eq. (8 20), the change in strain energy resulting from the virtual displacement is

$$\delta U = \int_0^l EA\frac{\partial u}{\partial x}\frac{\partial(\delta u)}{\partial x} dx \tag{8 21}$$

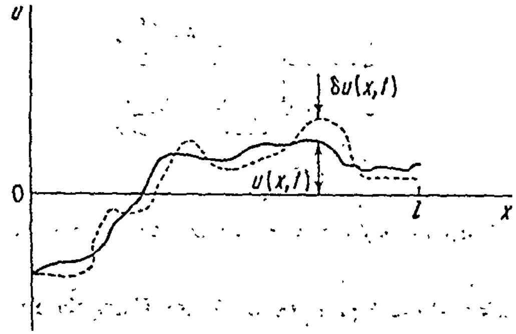


Fig. 8-5 *Longitudinal virtual displacement.*

Integrating by parts,

$$\delta U = EA\,\frac{\partial u}{\partial x}\,\delta u\,\bigg|_0^l - \int_0^l \frac{\partial}{\partial x}\left(EA\,\frac{\partial u}{\partial x}\right)\delta u\,dx$$

The work done by the elastic forces in the rod is given by the negative of the change in the strain energy. Then

$$\begin{aligned}\delta W_{el} &= -\delta U \\ &= -EA\,\frac{\partial u}{\partial x}\,\delta u\,\bigg|_0^l + \int_0^l \frac{\partial}{\partial x}\left(EA\,\frac{\partial u}{\partial x}\right)\delta u\,dx\end{aligned} \tag{8.22}$$

The virtual work done by the inertia forces on the differential element is given by $-\mu\,\dfrac{\partial^2 u}{\partial t^2}\,\delta u\,dx$. Summing by integrating over the bar

$$\delta W_{in} = -\int_0^l \mu\,\frac{\partial^2 u}{\partial t^2}\,\delta u\,dx \tag{8.23}$$

The distributed external force f_{ex} will do work on the bar in a virtual displacement. If there is a virtual displacement at the ends, the external forces acting at the ends will also do work. We can write

$$\delta W_{ex} = F_{ex}\,\delta u\,\bigg|_0^l + \int_0^l f_{ex}\,\delta u\,dx \tag{8.24}$$

From Eqs. (8.21) through (8.24)

$$\begin{aligned}\delta W &= \delta W_{el} + \delta W_{in} + \delta W_{ex} \\ &= \left(F_{ex} - EA\,\frac{\partial u}{\partial x}\right)\delta u\,\bigg|_0^l \\ &\quad + \int_0^l \left[\frac{\partial}{\partial x}\left(EA\,\frac{\partial u}{\partial x}\right) - \mu\,\frac{\partial^2 u}{\partial t^2} + f_{ex}\right]\delta u\,dx\end{aligned} \tag{8.25}$$

According to the principle of virtual displacements, $\delta W = 0$. Let us examine the integrated terms on the right-hand side of Eq. (8.25). If the displacement

u at the ends is specified in the problem, the virtual displacements δu at the ends are required to be zero. If instead the external force F_{ex} is prescribed at the ends, we will require $F_{ex} = EA\dfrac{\partial u}{\partial x}$. In either case the integrated terms are zero. Thus the integral on the right-hand side of Eq. (8.25) must be zero Since the virtual displacement δu is arbitrary, the integrand must be zero for all x and t. Then

$$\frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) - \mu\frac{\partial^2 u}{\partial t^2} + f_{ex} = 0 \tag{8.26}$$

which is the differential equation for the longitudinal vibration of the rod, given earlier by Eq (8.3).

The energy method of derivation serves to clarify the required boundary conditions as well as to produce the equation of motion. We can of course apply the method outlined to the other problems considered in Sec. 8.1.

8.3 The Wave Solution

As an example of the wave solution, let us obtain the solution for the free longitudinal motion of an elastic rod For a free motion, the distributed external force $f_{ex} = 0$ and the prescribed displacement or force at each of the ends must be zero. The mass per unit length of the rod is given by $\mu = \rho A$ in which ρ represents the mass density of the material. If the rod is uniform, the equation of motion from Eq (8 4) is

$$u'' - \frac{\rho}{E}\ddot{u} = 0 \tag{8.27}$$

As can be verified by substitution, we can write the general solution as

$$u = u_1(x - ct) + u_2(x + ct) \tag{8.28}$$

in which u_1 and u_2 are arbitrary functions of $x - ct$ and $x + ct$ respectively The quantity c is a constant

Consider the pair of positions x_1, x_2 and times t_1, t_2 which are related by $x_1 - ct_1 = x_2 - ct_2$ The displacement u_1 at x_1 and t_1 will evidently be identical with that at x_2 and t_2 In the period of time $t_2 - t_1$, the displacement will have propagated through the distance $x_2 - x_1 = c(t_2 - t_1)$ Thus the constant c represents the speed of wave propagation As shown in Fig. 8-6, the function u_1 represents a displacement pattern which is propagating in the positive direction at the speed c Similarly it can be shown that the function u_2 represents a displacement pattern which is propagating in the negative direction, again with speed c.

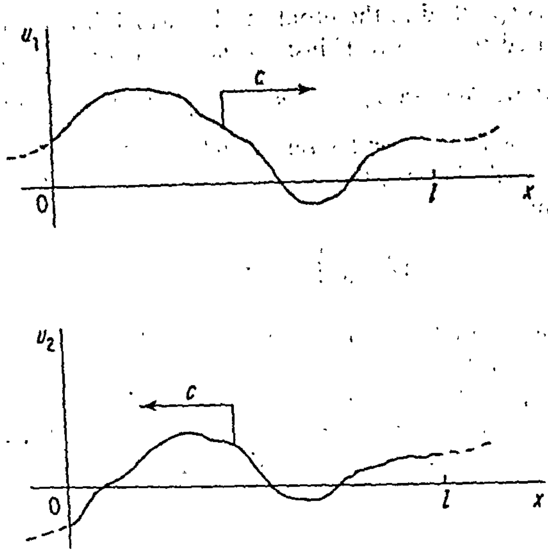


Fig. 8-6 *The wave solution for the longitudinal displacement of the rod.*

Differentiation of the functions u_1 and u_2 leads to

$$u_1'' = \frac{d^2u_1}{d(x-ct)^2}$$

$$\ddot{u}_1 = c^2\frac{d^2u_1}{d(x-ct)^2}$$

$$u_2'' = \frac{d^2u_2}{d(x+ct)^2}$$

$$\ddot{u}_2 = c^2\frac{d^2u_2}{d(x+ct)^2}$$

Substitution of the results for either u_1 or u_2 into Eq. (8.27) permits the evaluation of c, given by

$$c = \sqrt{\frac{E}{\rho}} \tag{8.29}$$

The functions u_1 and u_2 are determined by the initial conditions and the end conditions.

For a free torsional motion of a rod, the distributed external torque $\tau_{ex} = 0$ and the specified rotation or torque at each of the ends must be zero. If the rod is round, J represents the polar moment of inertia of the cross-sectional area. We can write the mass moment of inertia per unit length as $I = \rho J$.

Assuming the rod to be uniform, we can write the equation of motion, Eq. (8.10), as

$$\theta'' - \frac{\rho}{G}\ddot{\theta} = 0 \tag{8.30}$$

Since the equations of motion for longitudinal and torsional motion, Eqs. (8.27) and (8.30), have the same form, the solutions will have the same form. The general solution for torsional motion is

$$\theta = \theta_1(x - ct) + \theta_2(x + ct) \tag{8.31}$$

in which θ_1 and θ_2 represent displacement patterns in rotation which propagate in the positive and negative directions at the speed c. From Eq. (8.29), the speed of propagation c is

$$c = \sqrt{\frac{G}{\rho}} \tag{8.32}$$

We have looked at the simplest examples of wave solutions. Generally a wave solution is much more difficult to obtain. Further, the character of the solution is usually more complex. Wave patterns having a constant shape, such as those of Eqs. (8.28) and (8.31), are referred to as nondispersive waves. Often the shapes of the wave patterns vary with time and the waves are called dispersive waves.

The equation of motion for the flexural motion of a beam, Eq (8.17), does not have a wave solution. Having neglected shear deformation, we have effectively assumed the shear stiffness to be infinite. As a result, the equation of motion predicts infinite propagation speeds. We can modify the equation of motion by allowing shear deformation of the beam However, the wave solution for this case is quite difficult to obtain and is dispersive in character.

In general, the wave solution is most useful for simple systems and for excitation of very short duration. The method becomes unwieldy when applied to the usual engineering structure In practical engineering analysis, we will find the vibration solution to be the most useful. Thus we will confine ourselves after this to consideration of the vibration solution.

EXAMPLE *8.1*

For a uniform steel rod, we can approximate the physical properties by

$$E = 30 \times 10^6 \text{ lb/in.}^2$$
$$G = 10 \times 10^6 \text{ lb/in.}^2$$
$$\rho g = 490 \text{ lb/ft}^3$$

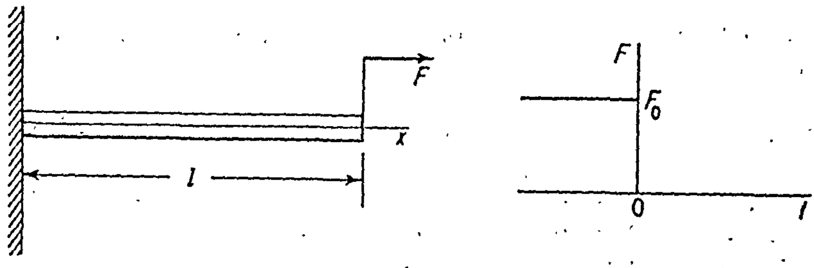


Fig. 8-7

From Eq. (8.29), the longitudinal wave speed is given by

$$c = \sqrt{\frac{30 \times 10^6 \times 144 \times 32.2}{490}}$$
$$= 16{,}850 \text{ ft/sec}$$

Using Eq. (8.32), the torsional wave speed for a round bar is given by

$$c = \sqrt{\frac{10 \times 10^6 \times 144 \times 32.2}{490}}$$
$$= 9720 \text{ ft/sec}$$

Example 8.2

A uniform rod is stretched by an axial force as shown in Fig. 8-7. If the force is suddenly relaxed, elastic waves will propagate through the rod. We can write the initial conditions, for $0 \leq x \leq l$, as

$$u(x,0) = \frac{F_0 l}{EA} \cdot \frac{x}{l}$$
$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0$$

and the end conditions, for $t > 0$, as

$$u(0,t) = 0$$
$$F_{ex}(l,t) = EAu'(x,t)|_{x=l} = 0$$

The ranges of values taken by the arguments of the functions u_1 and u_2 of Eq. (8.28) are

$$-\infty < x - ct < l$$
$$0 < x + ct < \infty$$

It can be shown that the functions u_1 and u_2 shown in Fig. 8-8 will satisfy the initial and end conditions we have specified. In Fig. 8-9, the displacement

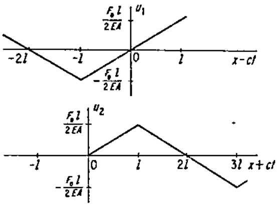


Fig. 8-8 *Solutions for $u_1(x - ct)$ and $u_2(x + ct)$*

patterns are shown for several instants of time following the relaxation of the end force. As shown, the displacement pattern returns to the original pattern at $t = \frac{4l}{c}$, the time required for a wave to propagate a distance $4l$. Thus the motion repeats itself with a period $\tau = \frac{4l}{c}$.

It is of interest to examine the propagation of the internal force in the bar, given by $F = EAu'$. We can represent the force by

$$F = F_1(x - ct) + F_2(x + ct)$$

where

$$F_1 = EA\frac{du_1}{d(x - ct)}$$
$$F_2 = EA\frac{du_2}{d(x + ct)}$$

In Fig 8-10, the internal force patterns are shown for several instants of time during a full period $\tau = \frac{4l}{c}$. From $t = 0$ to $t = \frac{l}{c}$ and from $t = \frac{2l}{c}$ to $t = \frac{3l}{c}$, an unloading wave propagates from the free end reducing the internal force to zero. From $t = \frac{l}{c}$ to $t = \frac{2l}{c}$, a compressive loading wave propagates from the fixed end resulting in a compressive force F_0. From $t = \frac{3l}{c}$ to $t = \frac{4l}{c}$, a tensile loading wave propagates from the fixed end resulting in a tensile force F_0.

Except for the rigid-body motion, each of the solutions represents a vibration having a shape $U(x)$ and a frequency ω. Any multiple of $U(x)$ serves to describe the shape of the motion. It is convenient to normalize each of the shapes in some arbitrary way, usually by setting the displacement at a selected point to unity. We will refer to each of the resulting functions as a normal mode shape, represented by $\phi(x)$ It is customary to arrange the values of the natural frequencies in the order of increasing magnitude, as given by $\omega_1 < \omega_2 < \cdots$. The solution for the motion in the ith normal mode of vibration can be written as

$$u(x,t) = \phi_i(x)q_i(t) \tag{8.39}$$

in which

$$q_i(t) = C_i \sin \omega_i t + C_i' \cos \omega_i t \tag{8.40}$$

For a rigid-body mode

$$q_i(t) = C_i + C_i' t \tag{8.41}$$

We will refer to $q_i(t)$ as the ith normal coordinate. Superimposing the motions in all of the normal modes, we can write the general solution for a free vibration as

$$u(x,t) = \sum_{i=1}^{\infty} \phi_i(x)q_i(t) \tag{8.42}$$

The arbitrary constants are determined by the initial conditions on the displacement and velocity.

Let us consider the longitudinal vibrations of a uniform rod. The first of Eq (8.35) becomes

$$U'' + \frac{\mu\omega^2}{EA} U = 0 \tag{8.43}$$

We can write the general solution for $U(x)$ as

$$U(x) = B_1 \sin \omega\sqrt{\frac{\mu}{EA}}\, x + B_2 \cos \omega\sqrt{\frac{\mu}{EA}}\, x \tag{8.44}$$

Application of the end conditions, Eqs. (8.36), leads to the solution for the natural frequencies ω and the corresponding amplitude ratios B_1/B_2 Knowledge of the amplitude ratios along with a suitable method for normalizing $U(x)$ permits us to write the normal mode shapes $\phi(x)$.

For the free torsional vibrations of a rod, the equation of motion from Eq. (8.10) is

$$(GJ\theta')' - I\ddot{\theta} = 0 \tag{8.45}$$

Following the method of separation of variables, our trial solution will have the form

$$\theta(x,t) = \Theta(x)T(t) \tag{8.46}$$

8.4 The Vibration Solution

(a) *Longitudinal or Torsional Vibrations of a Rod*

From Eq. (8.4), the equation of motion for the free longitudinal vibrations of a rod is

$$(EAu')' - \mu\ddot{u} = 0 \tag{8.33}$$

The vibration solution results from the use of the method of separation of variables. Let us try a solution

$$u(x,t) = U(x)T(t) \tag{8.34}$$

Substitution of the trial solution into the equation of motion leads to

$$(EAU')'T - \mu U\ddot{T} = 0$$

or to

$$\frac{(EAU')'}{\mu U} = \frac{\ddot{T}}{T}$$

The left-hand side of this equation is a function of x only and the right-hand side is a function of t only. Since the equation must hold for all x and all t, each side must be equal to a constant. Choosing the constant as $-\omega^2$, we can write

$$\begin{aligned}(EAU')' + \mu\omega^2 U &= 0\\ \ddot{T} + \omega^2 T &= 0\end{aligned} \tag{8.35}$$

From Eq. (8.6), the end conditions for fixed or free ends require that

$$\begin{aligned}U(0) = 0 \quad &\text{or} \quad EAU'(x)|_{x=0} = 0\\ U(l) = 0 \quad &\text{or} \quad EAU'(x)|_{x=l} = 0\end{aligned} \tag{8.36}$$

We will find that there are an infinite number of solutions for $U(x)$ which satisfy the first of Eqs. (8.35) and the end conditions. Corresponding to each $U(x)$ there is a definite value for ω^2. The quantities ω^2 and $U(x)$ are referred to as the eigenvalues and eigenfunctions of the system. Turning to the second of Eqs. (8.35), we can write the solution

$$T = C\sin\omega t + C'\cos\omega t \tag{8.37}$$

in which C and C' are arbitrary constants. If the system has a rigid-body degree of freedom, our experience should tell us that the corresponding eigenvalue ω^2 will be zero. For this case, the solution for the second of Eqs. (8.35) is

$$T = C + C't \tag{8.38}$$

where C and C' are again arbitrary constants.

Except for the rigid-body motion, each of the solutions represents a vibration having a shape $U(x)$ and a frequency ω. Any multiple of $U(x)$ serves to describe the shape of the motion. It is convenient to normalize each of the shapes in some arbitrary way, usually by setting the displacement at a selected point to unity. We will refer to each of the resulting functions as a normal mode shape, represented by $\phi(x)$. It is customary to arrange the values of the natural frequencies in the order of increasing magnitude, as given by $\omega_1 < \omega_2 < \cdots$. The solution for the motion in the ith normal mode of vibration can be written as

$$u(x,t) = \phi_i(x)q_i(t) \tag{8.39}$$

in which

$$q_i(t) = C_i \sin \omega_i t + C_i' \cos \omega_i t \tag{8.40}$$

For a rigid-body mode

$$q_i(t) = C_i + C_i' t \tag{8.41}$$

We will refer to $q_i(t)$ as the ith normal coordinate. Superimposing the motions in all of the normal modes, we can write the general solution for a free vibration as

$$u(x,t) = \sum_{i=1}^{\infty} \phi_i(x)q_i(t) \tag{8.42}$$

The arbitrary constants are determined by the initial conditions on the displacement and velocity.

Let us consider the longitudinal vibrations of a uniform rod The first of Eq. (8.35) becomes

$$U'' + \frac{\mu\omega^2}{EA} U = 0 \tag{8.43}$$

We can write the general solution for $U(x)$ as

$$U(x) = B_1 \sin \omega\sqrt{\frac{\mu}{EA}}\, x + B_2 \cos \omega\sqrt{\frac{\mu}{EA}}\, x \tag{8.44}$$

Application of the end conditions, Eqs (8.36), leads to the solution for the natural frequencies ω and the corresponding amplitude ratios B_1/B_2 Knowledge of the amplitude ratios along with a suitable method for normalizing $U(x)$ permits us to write the normal mode shapes $\phi(x)$

For the free torsional vibrations of a rod, the equation of motion from Eq. (8.10) is

$$(GJ\theta')' - I\ddot{\theta} = 0 \tag{8.45}$$

Following the method of separation of variables, our trial solution will have the form

$$\theta(x,t) = \Theta(x)T(t) \tag{8.46}$$

Substitution of the trial solution into the equation of motion leads to the pair of equations

$$\begin{gathered}(GJ\Theta')' + I\omega^2\Theta = 0 \\ \ddot{T} + \omega^2 T = 0\end{gathered} \tag{8.47}$$

From Eqs. (8.12), the end conditions are

$$\begin{aligned}\Theta(0) &= 0 \quad \text{or} \quad GJ\Theta'(x)|_{x=0} = 0 \\ \Theta(l) &= 0 \quad \text{or} \quad GJ\Theta'(x)|_{x=l} = 0\end{aligned} \tag{8.48}$$

There are an infinite number of solutions for the eigenfunctions $\Theta(x)$ and eigenvalues ω^2 which will satisfy the first of Eqs. (8.47) and the end conditions. The normal mode shapes $\phi(x)$ result from normalizing the eigenfunctions. Then the general solution for the free torsional vibrations of the rod will have the form of Eq. (8.42).

For a uniform rod, the first of Eqs. (8.47) becomes

$$\Theta'' + \frac{I\omega^2}{GJ}\Theta = 0 \tag{8.49}$$

We can write the general solution for $\Theta(x)$ as

$$\Theta(x) = B_1 \sin \omega\sqrt{\frac{I}{GJ}}\,x + B_2 \cos \omega\sqrt{\frac{I}{GJ}}\,x \tag{8.50}$$

Introduction of the end conditions, Eqs. (8.48), permits the evaluation of the natural frequencies ω and the corresponding amplitude ratios B_1/B_2.

Example 8.3

Consider the free longitudinal vibrations of the uniform rod shown in Fig. 8-7. The end conditions are

$$\begin{aligned}U(0) &= 0 \\ EAU'(x)|_{x=l} &= 0\end{aligned}$$

Substitution of the boundary conditions into the general solution, Eq. (8.44), leads to

$$\begin{aligned}0 &= B_2 \\ 0 &= B_1\omega\sqrt{\frac{\mu}{EA}} \cos \omega\sqrt{\frac{\mu}{EA}}\,l\end{aligned}$$

For a nontrivial solution, $B_1 \neq 0$ and $\omega \neq 0$. We must require $\cos \omega\sqrt{\frac{\mu}{EA}}\,l$ to be zero, leading to the solution for the natural frequencies

$$\omega\sqrt{\frac{\mu}{EA}}\,l = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

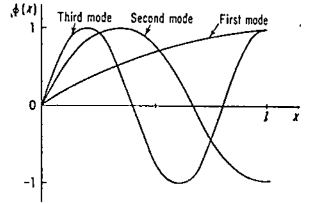


Fig. 8-11 *Normal mode shapes for the longitudinal vibrations of a uniform fixed-free rod.*

or

$$\omega_i = \frac{(2i - 1)\pi}{2}\sqrt{\frac{EA}{\mu l^2}} \qquad i = 1, 2, 3, \ldots$$

The corresponding shapes are

$$U_i(x) = B_{1i} \sin \frac{(2i - 1)\pi}{2}\frac{x}{l} \qquad i = 1, 2, 3, .$$

If we normalize the shapes by requiring the maximum displacement to be unity, we can write the normal mode shapes as

$$\phi_i(x) = \sin \frac{(2i - 1)\pi}{2}\frac{x}{l} \qquad i = 1, 2, 3, \ldots$$

The first three of the normal mode shapes are shown in Fig 8-11. We can write the general solution for a free vibration as

$$u(x,t) = \sum_{i=1}^{\infty} q_i(t) \sin \frac{(2i - 1)\pi}{2}\frac{x}{l}$$

EXAMPLE *8.4*

Let us examine the free torsional vibrations of a uniform rod free at both ends. From Eqs (8.48), we can write the end conditions as

$$GJ\Theta'(x)|_{x=0} = GJ\Theta'(x)|_{x=l} = 0$$

The general solution for $\Theta(x)$ is given by Eq. (8.50). Applying the end conditions

$$0 = \omega\sqrt{\frac{I}{GJ}}\, B_1$$

$$0 = \omega\sqrt{\frac{I}{GJ}}\left[B_1 \cos \omega\sqrt{\frac{I}{GJ}}\, l - B_2 \sin \omega\sqrt{\frac{I}{GJ}}\, l\right]$$

The solution $\omega = 0$ satisfies these equations and results in the shape function

$$\Theta(x) = B_2$$

Thus the first natural frequency is $\omega_1 = 0$. Requiring the maximum displacement to be unity, the corresponding normal mode shape is

$$\phi_1(x) = 1$$

The motion in the first normal mode is evidently rigid-body rotation.

For a solution with $\omega \neq 0$ and $B_2 \neq 0$, we must require $B_1 = 0$ and $\sin \omega\sqrt{\frac{I}{GJ}}\, l = 0$. It follows that there are an infinity of natural frequencies

$$\omega\sqrt{\frac{I}{GJ}}\, l = \pi, 2\pi, 3\pi, \ldots$$

or

$$\omega_i = (i - 1)\pi\sqrt{\frac{GJ}{Il^2}} \qquad i = 2, 3, \ldots$$

The corresponding shapes are

$$\Theta_i(x) = B_{2i} \cos \frac{(i-1)\pi}{l}\, x \qquad i = 2, 3, \ldots$$

Let us normalize the shapes by requiring the maximum rotation to be unity. Then the normal mode shapes are

$$\phi_i(x) = \cos\, (i - 1)\pi \frac{x}{l} \qquad i = 2, 3, \ldots$$

The expressions for the natural frequency and normal mode shape yield the results given earlier for the rigid-body mode if $i = 1$. The first three normal mode shapes are shown in Fig. 8-12. We can write the general solution for a free torsional vibration as

$$\theta(x,t) = \sum_{i=1}^{\infty} q_i(t) \cos\, (i - 1)\, \pi \frac{x}{l}$$

In this case $q_1(t)$ has the form of Eq. (8.41). The remaining normal coordinates $q_i(t)$ each describe a simple harmonic motion as given by Eq. (8.40).

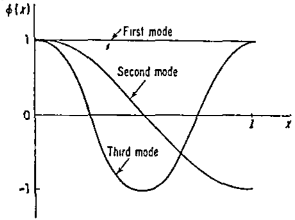


Fig. 8-12 *Normal mode shapes for the torsional vibrations of a uniform free-free rod*

(b) Flexural Vibrations of a Beam

We can write the equation for free flexural vibrations of a beam from Eq. (8.17) as

$$(EIw'')'' + \mu\ddot{w} = 0 \tag{8.51}$$

Applying the method of separation of variables, let us try a solution of the form

$$w(x,t) = W(x)T(t) \tag{8.52}$$

Substitution of the trial solution into the equation of motion results in

$$\frac{(EIW'')''}{\mu W} = -\frac{\ddot{T}}{T}$$

In order that this equation can be satisfied for all x and all t, it is necessary that each side must be equal to a constant If the constant is chosen as ω^2, we can write

$$\begin{aligned}(EIW'')'' - \mu\omega^2 W &= 0\\ \ddot{T} + \omega^2 T &= 0\end{aligned} \tag{8.53}$$

Using Eq. (8.19), we can write the end conditions for a free vibration of the beam as

$$\begin{aligned} W(0) = 0 \quad &\text{or} \quad [EIW''(x)]'|_{x=0} = 0\\ W'(x)|_{x=0} = 0 \quad &\text{or} \quad EIW''(x)|_{x=0} = 0\\ W(l) = 0 \quad &\text{or} \quad [EIW''(x)]'|_{x=l} = 0\\ W'(x)|_{x=l} = 0 \quad &\text{or} \quad EIW''(x)|_{x=l} = 0 \end{aligned} \tag{8.54}$$

There are an infinite number of solutions for the eigenfunctions $W(x)$ and corresponding eigenvalues ω^2 which will satisfy the first of Eqs. (8.53) and the end conditions. The time function T will describe simple harmonic motion, as given by Eq (8.37). The eigenvalue ω^2 corresponding to a rigid-

body degree of freedom will be zero. For this case, the time function T is given by Eq. (8.38). It is convenient to normalize the eigenfunctions $W(x)$, leading to the normal mode shapes, which we will identify as $\phi(x)$. Knowing the ith natural frequency ω_i and the ith normal mode shape $\phi_i(x)$, the solution for the motion is given by Eqs. (8.39) through (8.41). The general solution for the free flexural vibrations has the form of Eq. (8.42).

For a uniform beam, the first of Eqs. (8.53) becomes

$$W'''' - \frac{\mu\omega^2}{EI} W = 0 \tag{8.55}$$

The general solution for $W(x)$ is

$$\begin{aligned} W(x) = B_1 \sin\sqrt[4]{\frac{\mu\omega^2}{EI}}\, x + B_2 \cos\sqrt[4]{\frac{\mu\omega^2}{EI}}\, x \\ + B_3 \sinh\sqrt[4]{\frac{\mu\omega^2}{EI}}\, x + B_4 \cosh\sqrt[4]{\frac{\mu\omega^2}{EI}}\, x \end{aligned} \tag{8.56}$$

Introduction of the four end conditions from Eq. (8.54) into the general solution permits the evaluation of the natural frequencies ω. Further, three amplitude ratios relating the four constants B_1, B_2, B_3, B_4 can be obtained. Applying a suitable normalizing condition, we can write the normal mode shapes $\phi(x)$. In Appendix C, the results are given for the natural frequencies and normal mode shapes for uniform beams having several common combinations of end conditions. For more complete data, consult Ref. 19.

EXAMPLE 8.5

Consider the free flexural vibrations of a simply supported uniform beam. We can write the end conditions from Eq. (8.54) as

$$W(0) = W(l) = 0$$
$$EIW''(x)|_{x=0} = EIW''(x)|_{x=l} = 0$$

Substitution of the end conditions into the general solution for $W(x)$, Eq. (8-56), leads to

$$0 = B_2 + B_4$$
$$\begin{aligned} 0 = B_1 \sin\sqrt[4]{\frac{\mu\omega^2}{EI}}\, l + B_2 \cos\sqrt[4]{\frac{\mu\omega^2}{EI}}\, l \\ + B_3 \sinh\sqrt[4]{\frac{\mu\omega^2}{EI}}\, l + B_4 \cosh\sqrt[4]{\frac{\mu\omega^2}{EI}}\, l \end{aligned}$$
$$0 = B_2 - B_4$$
$$\begin{aligned} 0 = B_1 \sin\sqrt[4]{\frac{\mu\omega^2}{EI}}\, l + B_2 \cos\sqrt[4]{\frac{\mu\omega^2}{EI}}\, l \\ - B_3 \sinh\sqrt[4]{\frac{\mu\omega^2}{EI}}\, l - B_4 \cosh\sqrt[4]{\frac{\mu\omega^2}{EI}}\, l \end{aligned}$$

We can conclude from the first and third equations that $B_2 = B_4 = 0$.

Addition and subtraction of the second and fourth equations leads to

$$B_1 \sin \sqrt[4]{\frac{\mu\omega^2}{EA}}\, l = 0$$

$$B_3 \sinh \sqrt[4]{\frac{\mu\omega^2}{EA}}\, l = 0$$

The solution $\omega = 0$ leads to $W(x) = 0$ and is of no interest. For $\omega \neq 0$, the value for $\sinh \sqrt[4]{\frac{\mu\omega^2}{EA}}\, l$ cannot be zero and we must require $B_3 = 0$. For $B_1 \neq 0$, it is necessary that $\sin \sqrt[4]{\frac{\mu\omega^2}{EI}}\, l = 0$. As a result

$$\sqrt[4]{\frac{\mu\omega^2}{EI}}\, l = \pi, 2\pi, 3\pi, \ldots$$

and the natural frequencies are

$$\omega_i = (i\pi)^2 \sqrt{\frac{EI}{\mu l^4}} \qquad i = 1, 2, 3, \ldots$$

We can write the corresponding shapes as

$$W_i(x) = B_{1i} \sin \frac{i\pi x}{l} \qquad i = 1, 2, 3, \ldots$$

Setting the maximum amplitude to unity, the normal mode shapes become

$$\phi_i(x) = \sin \frac{i\pi x}{l} \qquad i = 1, 2, 3, \ldots$$

The first three normal shapes are shown in Fig 8-13. The general solution

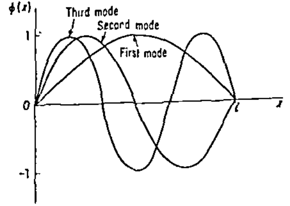


Fig. 8-13 *Normal mode shapes for the flexural vibrations of a uniform simply supported beam.*

for the free vibrations is given by

$$w(x,t) = \sum_{i=1}^{\infty} q_i(t) \sin \frac{i\pi x}{l}$$

in which the solution for the normal coordinates $q_i(t)$ is given by Eq. (8.40).

8.5 Orthogonality of the Normal Modes

(a) *Longitudinal or Torsional Vibrations of a Rod*

For the free longitudinal vibrations of a rod, the natural frequencies and normal mode shapes represent solutions of Eq. (8.35). For the ith and jth normal modes, we can write

$$\begin{aligned} (EA\phi_i')' &= -\mu\omega_i^2\phi_i \\ (EA\phi_j')' &= -\mu\omega_j^2\phi_j \end{aligned} \tag{8.57}$$

Let us premultiply the two equations by ϕ_j and ϕ_i respectively, and integrate the results with respect to x over the length of the rod.

$$\begin{aligned} \int_0^l (EA\phi_i')'\phi_j \, dx &= -\omega_i^2 \int_0^l \mu\phi_i\phi_j \, dx \\ \int_0^l (EA\phi_j')'\phi_i \, dx &= -\omega_j^2 \int_0^l \mu\phi_i\phi_j \, dx \end{aligned}$$

Integrating the left-hand sides of the two equations by parts, we can write the results as

$$\begin{aligned} EA\phi_i'\phi_j \Big|_0^l - \int_0^l EA\phi_i'\phi_j' \, dx &= -\omega_i^2 \int_0^l \mu\phi_i\phi_j \, dx \\ EA\phi_j'\phi_i \Big|_0^l - \int_0^l EA\phi_i'\phi_j' \, dx &= -\omega_j^2 \int_0^l \mu\phi_i\phi_j \, dx \end{aligned} \tag{8.58}$$

Since the mode shapes must satisfy the free vibration boundary conditions, given by $\phi = 0$ or $EA\phi' = 0$, the integrated terms on the left-hand side are zero. Subtraction of the first of Eqs. (8.58) from the second leads to

$$0 = (\omega_i^2 - \omega_j^2) \int_0^l \mu\phi_i\phi_j \, dx \tag{8.59}$$

If the eigenvalues ω_i^2 and ω_j^2 are unequal, we can conclude from Eqs. (8.58) and (8.59) that

$$\begin{aligned} \int^l \mu\phi_i\phi_j \, dx &= 0 \\ &\quad 0 \end{aligned} \tag{8.60}$$

If the eigenvalues are all distinct, Eqs. (8.60) require that $i \neq j$. The two equations describe the property of orthogonality of the normal modes. If there are repeated eigenvalues, each of the associated normal modes will be orthogonal to any normal mode associated with a different eigenvalue. Although the normal modes associated with repeated eigenvalues need not be orthogonal to each other, it is always possible to require them to be.

We can show that the physical significance of the orthogonality relations, Eqs (8.60), is that the ith and jth normal modes are uncoupled, both inertially and elastically. The acceleration of the rod resulting from motion in the jth normal mode is given by $\ddot{u} = \phi_j \ddot{q}_j$. Then the inertia force acting on an element of the rod is given by $-\mu\phi_j\ddot{q}_j\, dx$. A virtual displacement of the rod in the ith normal coordinate is described by $\delta u = \phi_i\, \delta q_i$. We can write the work done by the inertia forces in the virtual displacement as

$$\delta W_{in} = -\ddot{q}_j\, \delta q_i \int_0^l \mu\phi_i\phi_j\, dx$$

From the first of the orthogonality relations, Eq (8.60), the work done is zero, provided that $i \neq j$. Evidently the inertia forces resulting from motion of one of the normal modes will have no influence on the motion of one of the other normal modes. As a result, the normal modes are uncoupled inertially. Following a similar procedure, we can show with the aid of the second of the orthogonality relations, Eqs (8.60), that the normal modes are uncoupled elastically.

Let us write the equations of motion for a free longitudinal vibration of the rod in terms of the normal coordinates. The kinetic energy of the rod in motion is

$$T = \frac{1}{2}\int_0^l \mu\dot{u}^2\, dx \tag{8.61}$$

or, in view of Eq. (8.42)

$$\begin{aligned} T &= \frac{1}{2}\int_0^l \mu\left(\sum_{i=1}^{\infty} \phi_i\dot{q}_i\right)\left(\sum_{j=1}^{\infty} \phi_j\dot{q}_j\right) dx \\ &= \frac{1}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \dot{q}_i\dot{q}_j \int_0^l \mu\phi_i\phi_j\, dx \end{aligned}$$

Using the first of the orthogonality relations, Eqs. (8.60), it is evident that the only nonzero terms in the kinetic energy expression are those for which $i = j$. Then

$$T = \frac{1}{2}\sum_{i=1}^{\infty} \dot{q}_i^2 \int_0^l \mu\phi_i^2\, dx \tag{8.62}$$

Let us define the generalized masses associated with the normal coordinates by

$$M_i = \int_0^l \mu\phi_i^2\,dx \qquad i = 1, 2, 3, \ldots \tag{8.63}$$

We can write the kinetic energy in the form

$$T = \frac{1}{2}\sum_{i=1}^{\infty} M_i\dot{q}_i^2 \tag{8.64}$$

The generalized inertia forces are given by

$$Q_{i,in} = -\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = -M_i\ddot{q}_i \tag{8.65}$$

The strain energy stored in a deformed elastic rod is

$$U = \frac{1}{2}\int_0^l EAu'^2\,dx$$

From Eq. (8.42)

$$\begin{aligned} U &= \frac{1}{2}\int_0^l EA\left(\sum_{i=1}^{\infty}\phi_i' q_i\right)\left(\sum_{j=1}^{\infty}\phi_j' q_j\right)dx \\ &= \frac{1}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} q_i q_j \int_0^l EA\phi_i'\phi_j'\,dx \end{aligned}$$

As a result of the second of the orthogonality relations, Eqs. (8.60), we can simplify this expression to

$$U = \frac{1}{2}\sum_{i=1}^{\infty} q_i^2 \int_0^l EA\phi_i'^2\,dx \tag{8.66}$$

The generalized stiffnesses associated with the normal coordinates are defined by

$$K_i = \int_0^l EA\phi_i'^2\,dx \tag{8.67}$$

Then the strain energy is

$$U = \frac{1}{2}\sum_{i=1}^{\infty} K_i q_i^2 \tag{8.68}$$

The generalized elastic forces are

$$Q_{i,el} = -\frac{\partial U}{\partial q_i} = -K_i q_i \qquad i = 1, 2, \ldots \tag{8.69}$$

For free longitudinal vibrations, we can write the equations of motion as

$$\begin{aligned} \sum Q_i &= Q_{i,in} + Q_{i,el} = 0 \\ &= -M_i\ddot{q}_i - K_i q_i = 0 \qquad i = 1, 2, \ldots \end{aligned} \tag{8.70}$$

The normal coordinates q_i must satisfy the equations of motion and the initial conditions. Using Eq. (8 42), we can write the initial conditions on the displacements and velocities as

$$
\begin{aligned}
u_0(x) &= \sum_{i=1}^{\infty} \phi_i(x) q_i(0) \\
\dot{u}_0(x) &= \sum_{i=1}^{\infty} \phi_i(x) \dot{q}_i(0)
\end{aligned}
\tag{8.71}
$$

We can determine the initial conditions on the normal coordinates using either of the orthogonality relations, Eqs. (8.60). Multiplication of Eqs. (8.71) by $\mu(x)\phi_j(x)$ and integration with respect to x over the length of the bar leads to

$$
\begin{aligned}
\int_0^l \mu u_0 \phi_j \, dx &= \sum_{i=1}^{\infty} q_i(0) \int_0^l \mu \phi_i \phi_j \, dx \\
\int_0^l \mu \dot{u}_0 \phi_j \, dx &= \sum_{i=1}^{\infty} \dot{q}_i(0) \int_0^l \mu \phi_i \phi_j \, dx
\end{aligned}
$$

Using the first of Eqs. (8.60), we can write the result

$$
\begin{aligned}
q_j(0) &= \frac{1}{M_j} \int_0^l \mu u_0 \phi_j \, dx \\
\dot{q}_j(0) &= \frac{1}{M_j} \int_0^l \mu \dot{u}_0 \phi_j \, dx
\end{aligned}
\tag{8.72}
$$

in which the generalized mass M_j is defined by Eq (8 63) Let us differentiate Eqs. (8.71) once with respect to x, multiply them by $EA(x) \times \phi_j'(x)$, and integrate them with respect to x over the length of the rod

$$
\begin{aligned}
\int_0^l EA u_0' \phi_j' \, dx &= \sum_{i=1}^{\infty} q_i(0) \int_0^l EA \phi_i' \phi_j' \, dx \\
\int_0^l EA \dot{u}_0' \phi_j' \, dx &= \sum_{i=1}^{\infty} \dot{q}_i(0) \int_0^l EA \phi_i' \phi_j' \, dx
\end{aligned}
$$

Using the second of Eqs. (8.60), we can write

$$
\begin{aligned}
q_j(0) &= \frac{1}{K_j} \int_0^l EA u_0' \phi_j' \, dx \\
\dot{q}_j(0) &= \frac{1}{K_j} \int_0^l EA \dot{u}_0' \phi_j' \, dx
\end{aligned}
\tag{8.73}
$$

The generalized stiffness K_j is defined by Eq (8.67) The solution for the free motion of the rod in the normal coordinates is given by Eqs. (8.40) and (8.41) We can determine the constants C_i and C_i' using either Eqs. (8 72) or (8.73). However, if there is rigid body translation, we will not be able to use Eqs (8 73) for $j = 1$ since $K_1 = 0$

The discussion above is appropriate for the free torsional vibrations of a rod. Referring to Eqs. (8.60), we can write the orthogonality relations for the torsional vibrations as

$$\begin{aligned} \int_0^l I\phi_i\phi_j\,dx &= 0 \\ \int_0^l GJ\phi_i'\phi_j'\,dx &= 0 \end{aligned} \tag{8.74}$$

For distinct eigenvalues, these relations hold provided $i \neq j$. The equations of motion for a free vibration have the form of Eq. (8.70) in which the generalized masses and stiffnesses are defined by

$$\begin{aligned} M_i &= \int_0^l I\phi_i^2\,dx \\ K_i &= \int_0^l GJ\phi_i'^2\,dx \end{aligned} \tag{8.75}$$

If $\theta_0(x)$ and $\dot{\theta}_0(x)$ represent the initial conditions on the rotation and rotational velocity, we can write the initial conditions on the normal coordinates from Eqs. (8.72) and (8.73). The results are given by

$$\begin{aligned} q_i(0) &= \frac{1}{M_i}\int_0^l I\theta_0\,\phi_i\,dx \\ \dot{q}_i(0) &= \frac{1}{M_i}\int_0^l I\dot{\theta}_0\phi_i\,dx \end{aligned} \tag{8.76}$$

or by

$$\begin{aligned} q_i(0) &= \frac{1}{K_i}\int_0^l GJ\theta_0'\dot{\phi}_i'\,dx \\ \dot{q}_i(0) &= \frac{1}{K_i}\int_0^l GJ\dot{\theta}_0'\phi_i'\,dx \end{aligned} \tag{8.77}$$

EXAMPLE 8.6

Let us obtain the solution for the free vibrations of the rod shown in Fig. 8-7. From Example 8.3, the general solution for the free vibrations of the rod is

$$u(x,t) = \sum_{i=1}^{\infty} q_i(t)\sin\frac{(2i-1)\pi}{2}\cdot\frac{x}{l}$$

We can write the initial conditions on the displacement and velocity as

$$\begin{aligned} \frac{F_0 l}{EA}\cdot\frac{x}{l} &= \sum_{i=1}^{\infty} q_i(0)\sin\frac{(2i-1)\pi}{2}\cdot\frac{x}{l} \\ 0 &= \sum_{i=1}^{\infty} \dot{q}_i(0)\sin\frac{(2i-1)\pi}{2}\cdot\frac{x}{l} \end{aligned}$$

Let us multiply each equation by $\mu \sin \frac{(2j-1)\pi}{2} \cdot \frac{x}{l}$ and integrate with respect to x over the length of the rod.

$$\int_0^l \frac{F_0 l}{EA} \cdot \frac{x}{l} \cdot \mu \sin \frac{(2j-1)\pi}{2} \cdot \frac{x}{l}\, dx$$
$$= \sum_{i=1}^{\infty} q_i(0) \int_0^l \mu \sin \frac{(2i-1)\pi}{2} \cdot \frac{x}{l} \sin \frac{(2j-1)\pi}{2} \cdot \frac{x}{l}\, dx$$
$$0 = \sum_{i=1}^{\infty} \dot{q}_i(0) \int_0^l \mu \sin \frac{(2i-1)\pi}{2}\, \frac{x}{l} \sin \frac{(2j-1)\pi}{2} \cdot \frac{x}{l}\, dx$$

From the first of the orthogonality relations, Eq. (8.60), the integrals on the right-hand side of the equations are zero if $i \neq j$. The generalized mass associated with the jth normal coordinate is given by Eq. (8.63) as

$$M_j = \int_0^l \mu \sin^2 \frac{(2j-1)\pi}{2}\, \frac{x}{l}\, dx = \frac{\mu l}{2}$$

The remaining integration results in

$$\int_0^l \frac{F_0 l}{EA}\, \frac{x}{l} \cdot \mu \sin \frac{(2j-1)\pi}{2}\, \frac{x}{l}\, dx = \frac{4(-1)^{j+1}}{(2j-1)^2\pi^2}\, \frac{F_0 l^2}{EA}\, \mu$$

Then we can write the initial conditions as

$$q_j(0) = \frac{8(-1)^{j+1}}{(2j-1)^2\pi^2} \cdot \frac{F_0 l}{EA}$$
$$\dot{q}_j(0) = 0$$

Substitution of the initial conditions into the general solution for the normal coordinates, given by

$$q_j(t) = C_j \sin \omega_j t + C_j' \cos \omega_j t$$

leads to

$$C_j = 0$$
$$C_j' = \frac{8(-1)^{j+1}}{(2j-1)^2\pi^2}\, \frac{F_0 l}{EA}$$

We can write the solution for the displacements as

$$u(x,t) = \frac{8}{\pi^2}\, \frac{F_0 l}{EA} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(2j-1)^2} \sin \frac{(2j-1)\pi}{2}\, \frac{x}{l} \cos \omega_j t$$

The solution for the axial force in the rod is

$$F(x,t) = EAu'(x,t)$$
$$= \frac{4}{\pi} F_0 \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{2j-1} \cos \frac{(2j-1)\pi}{2}\, \frac{x}{l} \cos \omega_j t$$

From the results of Example 8.3, the natural frequencies are

$$\omega_i = \frac{(2i-1)\pi}{2}\sqrt{\frac{EA}{\mu l^2}} \qquad i = 1, 2, 3, \ldots$$

Let us compare the vibration solution with the wave solution of Example 8.2 for the arbitrary instant of time $t = \frac{l}{2c}$. Recalling that $\mu = \rho A$ and that the wave speed $c = \sqrt{\frac{E}{\rho}}$, we can write for the first three modes that

$$\begin{aligned}
\omega_1 t &= \frac{\pi}{2}\sqrt{\frac{EA}{\rho A l^2}}\,\frac{l}{2}\sqrt{\frac{\rho}{E}} = \frac{\pi}{4}\\
\omega_2 t &= \frac{3\pi}{4}\\
\omega_3 t &= \frac{5\pi}{4}
\end{aligned}$$

Then the displacements and forces in the first three modes at the instant $t = \frac{l}{2c}$ are

$$\begin{aligned}
u &= \frac{4\sqrt{2}}{\pi^2}\,\frac{F_0 l}{EA}\left(\sin\frac{\pi x}{2l} + \frac{1}{9}\sin\frac{3\pi x}{2l} - \frac{1}{25}\sin\frac{5\pi x}{2l}\right)\\
F &= \frac{2\sqrt{2}}{\pi}F_0\left(\cos\frac{\pi x}{2l} + \frac{1}{3}\cos\frac{3\pi x}{2l} - \frac{1}{5}\cos\frac{5\pi x}{2l}\right)
\end{aligned}$$

For comparison, the wave solutions for $t = \frac{l}{2c}$ are given by Figs. 8-9 and 8-10. The displacement solution is quite convergent with mode number and the three-mode solution is close to that given by the exact wave solution. It is evident that the force solution is less convergent with the mode number. The three-mode force solution is shown in Fig. 8-14 along with the exact wave solution. As we add more modes to the vibration solution, the result will approach more and more closely the exact wave solution.

(b) *Flexural Vibrations of a Beam*

The natural frequencies and normal mode shapes for the free flexural vibrations of a beam must satisfy the first of Eqs. (8.53). For the ith and jth normal modes

$$\begin{aligned}
(EI\phi_i'')'' &= \mu\omega_i^2\phi_i\\
(EI\phi_j'')'' &= \mu\omega_j^2\phi_j
\end{aligned} \tag{8.78}$$

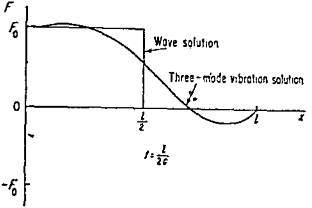


Fig. 8-14 *Comparison of the exact wave solution with the approximate vibration solution for the force in the rod.*

Let us premultiply the equations by ϕ_j and ϕ_i respectively and integrate the results with respect to x over the length of the beam.

$$\int_0^l (EI\phi_i'')''\phi_j\,dx = \omega_i^2 \int_0^l \mu\phi_i\phi_j\,dx$$
$$\int_0^l (EI\phi_j'')''\phi_i\,dx = \omega_j^2 \int_0^l \mu\phi_i\phi_j\,dx$$

Integration of the left-hand sides of the two equations by parts leads to

$$\begin{aligned}(EI\phi_i'')'\phi_j\Big|_0^l - EI\phi_i''\phi_j'\Big|_0^l + \int_0^l EI\phi_i''\phi_j''\,dx &= \omega_i^2 \int_0^l \mu\phi_i\phi_j\,dx \\ (EI\phi_j'')'\phi_i\Big|_0^l - EI\phi_j''\phi_i'\Big|_0^l + \int_0^l EI\phi_i''\phi_j''\,dx &= \omega_j^2 \int_0^l \mu\phi_i\phi_j\,dx\end{aligned} \tag{8.79}$$

The integrated terms are zero since the mode shapes must satisfy the free vibration end conditions. Subtraction of the first of Eqs (8.79) from the second results in

$$0 = (\omega_j^2 - \omega_i^2)\int_0^l \mu\phi_i\phi_j\,dx \tag{8.80}$$

If the eigenvalues are distinct and $i \neq j$, it is evident from Eqs (8.79) and (8.80) that

$$\begin{aligned}\int_0^l \mu\phi_i\phi_j\,dx &= 0 \\ \int_0^l EI\phi_i''\phi_j''\,dx &= 0\end{aligned} \tag{8.81}$$

These are the orthogonality relations for the normal modes.

The equations of motion for the free flexural vibrations of a beam will have the form of Eq. (8.70) in which the generalized masses and stiffnesses are defined by

$$M_i = \int_0^l \mu\phi_i^2\,dx$$
$$K_i = \int_0^l EI\phi_i''^2\,dx \tag{8.82}$$

Let $w_0(x)$ and $\dot{w}_0(x)$ represent the initial conditions on the transverse displacement and velocity of the beam. The initial conditions on the normal coordinates are given by

$$q_i(0) = \frac{1}{M_i}\int_0^l \mu w_0\phi_i\,dx$$
$$\dot{q}_i(0) = \frac{1}{M_i}\int_0^l \mu \dot{w}_0\phi_i\,dx \tag{8.83}$$

or by

$$q_i(0) = \frac{1}{K_i}\int_0^l EIw_0''\phi_i''\,dx$$
$$\dot{q}_i(0) = \frac{1}{K_i}\int_0^l EI\dot{w}_0''\phi_i''\,dx \tag{8.84}$$

This result was obtained using the orthogonality relations.

EXAMPLE 8.7

The elements of the simply supported uniform beam of Fig. 8-15 are initially given a uniform transverse velocity V as shown. From Example 8.5, the general solution for the free vibration of the beam is

$$w(x,t) = \sum_{i=1}^{\infty} q_i(t)\sin\frac{i\pi x}{l}$$

We can write the initial conditions on the displacement and velocity of the beam elements as

$$w_0(x) = 0 = \sum_{i=1}^{\infty} q_i(0)\sin\frac{i\pi x}{l}$$
$$\dot{w}_0(x) = V = \sum_{i=1}^{\infty} \dot{q}_i(0)\sin\frac{i\pi x}{l}$$

Let us multiply each equation by $\mu\sin\frac{j\pi x}{l}$ and integrate with respect to x over the length of the beam

$$0 = \sum_{i=1}^{\infty} q_i(0)\int_0^l \mu\sin\frac{i\pi x}{l}\sin\frac{j\pi x}{l}\,dx$$
$$\int_0^l V\mu\sin\frac{j\pi x}{l}\,dx = \sum_{i=1}^{\infty} \dot{q}_i(0)\int_0^l \mu\sin\frac{i\pi x}{l}\sin\frac{j\pi x}{l}\,dx$$

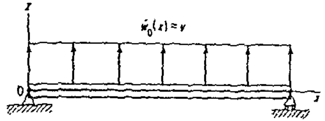


Fig. 8-15

From the first of the orthogonality relations, Eqs. (8 81), the integrals on the right-hand side of the equations are zero if $i \neq j$. The generalized masses associated with the normal coordinates are given by the first of Eqs. (8.82) as

$$M_j = \int_0^l \mu \sin^2 \frac{j\pi x}{l}\, dx = \frac{\mu l}{2}$$

Further,

$$\begin{aligned} \int_0^l V\mu \sin \frac{j\pi x}{l}\, dx &= \frac{2}{j\pi}\mu l V \qquad \text{for } j = 1, 3, 5, \ldots \\ &= 0 \qquad \text{for } j = 2, 4, 6, \ldots \end{aligned}$$

Then the initial conditions on the normal coordinates are

$$\begin{aligned} q_j(0) &= 0 \\ \dot{q}_j(0) &= \frac{4}{j\pi} V \qquad \text{for } j = 1, 3, 5, \\ &= 0 \qquad \text{for } j = 2, 4, 6, \ldots \end{aligned}$$

The general solution for each of the normal coordinates has the form

$$q_j(t) = C_j \sin \omega_j t + C_j' \cos \omega_j t$$

Applying the initial conditions

$$\begin{aligned} C_j &= \frac{4}{j\pi}\frac{V}{\omega_j} \qquad \text{for } j = 1, 3, 5, \ldots \\ &= 0 \qquad \text{for } j = 2, 4, 6, \ldots \\ C_j' &= 0 \end{aligned}$$

From Example 8.5, the natural frequencies are

$$\omega_j = (j\pi)^2 \sqrt{\frac{EI}{\mu l^4}} \qquad j = 1, 2, 3, \ldots$$

We can write the solution for the transverse displacements of the beam as

$$w(x,t) = 4V\sqrt{\frac{\mu l^4}{EI}} \sum_{i=1,3,5,}^{\infty} \frac{1}{(i\pi)^3} \sin \frac{i\pi x}{l} \sin \omega_i t$$

The bending moment in the beam is given by

$$M(x,t) = EIw'' = 4V\sqrt{\mu EI} \sum_{i=1,3,5,\ldots}^{\infty} \frac{1}{i\pi} \sin \frac{i\pi x}{l} \sin \omega_i t$$

Note that the displacements and moments in the modes vary with the mode number i as i^{-3} and i^{-1} respectively. Thus the displacement solution is very convergent with the mode number and only a few modes are needed for a close answer. Evidently, the moment solution is not very convergent and many more modes are needed in the solution.

8.6 Systems with Lumped Mass or Stiffness

(a) Longitudinal Vibrations of a Rod with a Lumped Mass

As an example of a continuous system having one lumped mass, consider the free longitudinal vibrations of the elastic rod shown in Fig. 8-16. A rigid body of mass m is attached to the end of the rod at $x = l$. For the system in motion, the equation of motion for the rod is given by Eq. (8.4). Referring to the free-body diagram of Fig. 8-16, we can write the end conditions as

$$\begin{aligned} u(0,t) &= 0 \\ F(l,t) &= EAu'(x,t)|_{x=l} = -m\ddot{u}(x,t)|_{x=l} \end{aligned} \tag{8.85}$$

Applying the method of separation of variables, the functions $U(x)$ and $T(t)$ must satisfy Eqs. (8.35) as before. Substitution of Eq. (8.34) into Eqs. (8.85) leads to the end conditions

$$\begin{aligned} U(0) &= 0 \\ EAU'(x)|_{x=l} - m\omega^2 U(l) &= 0 \end{aligned} \tag{8.86}$$

We can obtain the eigenfunctions $U(x)$ and the corresponding eigenvalues ω^2 by solving the first of Eqs. (8.35) along with the end conditions. By suitably normalizing the eigenfunctions, we can write the normal mode shapes $\phi(x)$. The general solution for the free vibrations in terms of the normal coordinates is given by Eq. (8.42) as before.

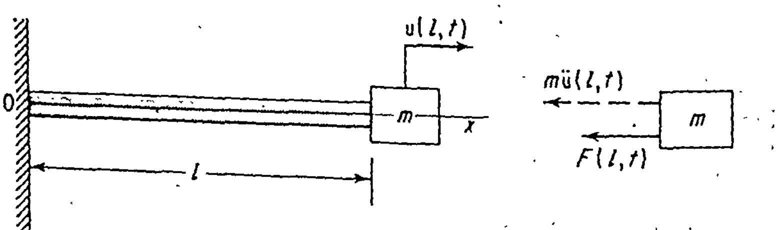


Fig. 8-16

We can establish the form of the orthogonality relations from the principle of virtual displacements. The inertia forces resulting from motion in the ith mode consist of the distributed force per unit length $-\mu\phi_i\ddot{q}_i$ and the concentrated force $-m\phi_i(l)\ddot{q}_i$ at the end of the rod. In a virtual displacement of the jth mode, represented by $\phi_j\,\delta q_j$, the work done is

$$\begin{aligned}\delta W_{in} &= \int_0^l (-\mu\phi_i\ddot{q}_i)(\phi_j\,\delta q_j)\,dx - m\phi_i(l)\ddot{q}_i\phi_j(l)\,\delta q_j \\ &= -\ddot{q}_i\,\delta q_j\left[\int_0^l \mu\phi_i\phi_j\,dx + m\phi_i(l)\phi_j(l)\right]\end{aligned}$$

Since the work done must be zero, we anticipate that one of the orthogonality relations will be the expression in brackets set to zero. The form of the other orthogonality relation can be determined by writing the virtual work done by the elastic forces. However, the addition of a lumped mass can have no effect on the expression for δW_{el} and we expect the second orthogonality relation to be unchanged.

The ith and jth eigenvalues and normal mode shapes must satisfy the second of the end conditions, Eqs (8.86), leading to

$$\begin{aligned}EA\phi_i'(l) &= \omega_i^2 m\phi_i(l)\\ EA\phi_j'(l) &= \omega_j^2 m\phi_j(l)\end{aligned}$$

Let us multiply these equations by $\phi_j(l)$ and $\phi_i(l)$ respectively and subtract them from the first and second of Eqs (8 58) respectively. This operation results in

$$\begin{aligned}EA\phi_i'(0)\phi_j(0) - \int_0^l EA\phi_i'\phi_j'\,dx &= -\omega_i^2\left[\int_0^l \mu\phi_i\phi_j\,dx + m\phi_i(l)\phi_j(l)\right]\\ EA\phi_j'(0)\phi_i(0) - \int_0^l EA\phi_i'\phi_j'\,dx &= -\omega_j^2\left[\int_0^l \mu\phi_i\phi_j\,dx + m\phi_i(l)\phi_j(l)\right]\end{aligned} \tag{8.87}$$

From the first of the end conditions, Eqs. (8 86), we can write $\phi_i(0) = \phi_j(0) = 0$. Since the integrated terms on the left-hand sides of Eqs (8.87) are zero, we can write the orthogonality relations as

$$\begin{aligned}\int_0^l \mu\phi_i\phi_j\,dx + m\phi_i(l)\phi_j(l) &= 0\\ \int_0^l EA\phi_i'\phi_j'\,dx &= 0\end{aligned} \tag{8.88}$$

provided the eigenvalues are distinct and $i \neq j$ Evidently the addition of a lumped mass adds a term to the first orthogonality relation, the one associated with the inertia forces. The second orthogonality relation, the one associated with the elastic forces, is unaffected

We can write the kinetic energy of the system of Fig. 8-16 as

$$T = \frac{1}{2}\int_0^l \mu\dot{u}^2\,dx + \tfrac{1}{2}m\dot{u}^2(l,t)$$

or, in terms of the normal coordinates, as

$$T = \frac{1}{2}\int_0^l \mu\left(\sum_{i=1}^{\infty} \phi_i \dot{q}_i\right)\left(\sum_{j=1}^{\infty} \phi_j \dot{q}_j\right) dx + \tfrac{1}{2}m\left(\sum_{i=1}^{\infty} \phi_i(l)\dot{q}_i\right)\left(\sum_{j=1}^{\infty} \phi_j(l)\dot{q}_j\right)$$

$$= \frac{1}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \dot{q}_i\dot{q}_j\left[\int_0^l \mu\phi_i\phi_j\, dx + m\phi_i(l)\phi_j(l)\right]$$

In view of the first of the orthogonality relations, Eq. (8.88), we can simplify the results to

$$T = \frac{1}{2}\sum_{i=1}^{\infty} M_i\dot{q}_i^2$$

in which the generalized masses are defined by

$$M_i = \int_0^l \mu\phi_i^2\, dx + m\phi_i^2(l) \tag{8.89}$$

As before, the generalized stiffnesses are defined by Eq. (8.67). The generalized inertia and elastic forces will have the form of Eqs. (8.65) and (8.69) and the equations of motion for a free vibration are given by Eq. (8.70).

Making use of the first of the orthogonality relations, Eqs. (8.88), we can write the initial conditions on the normal coordinates as

$$\begin{aligned} q_i(0) &= \frac{1}{M_i}\left[\int_0^l \mu u_0\phi_i\, dx + mu_0(l)\phi_i(l)\right] \\ \dot{q}_i(0) &= \frac{1}{M_i}\left[\int_0^l \mu\dot{u}_0\phi_i\, dx + m\dot{u}_0(l)\phi_i(l)\right] \end{aligned} \tag{8.90}$$

This result is obtained by following the general procedure which led to Eqs. (8.72). As an alternate, the initial conditions on the normal coordinates are given by Eqs. (8.73).

EXAMPLE 8.8

Suppose the rod of Fig. 8-16 is uniform. The general solution for the eigenfunctions $U(x)$ is given by Eq. (8.44). Substitution of the end conditions, Eqs. (8.86), into the general solution leads to

$$B_2 = 0$$

$$B_1\left[EA\omega\sqrt{\frac{\mu}{EA}}\cos\omega\sqrt{\frac{\mu}{EA}}\,l - m\omega^2\sin\omega\sqrt{\frac{\mu}{EA}}\,l\right] = 0$$

The solutions $B_1 = 0$ or $\omega = 0$ are trivial, since they lead to $U(x) = 0$. Setting the term in the brackets to zero

$$\omega\sqrt{\frac{\mu l^2}{EA}}\tan\omega\sqrt{\frac{\mu l^2}{EA}} = \frac{\mu l}{m}$$

Consider the special case in which the masses of the rod and of the rigid body are equal, as given by $\frac{\mu l}{m} = 1$. For this case a trial solution of the transcendental frequency equation leads to

$$\omega_1 = 0.860\sqrt{\frac{EA}{\mu l^2}}$$
$$\omega_2 = 3.42\sqrt{\frac{EA}{\mu l^2}}$$

Normalizing the eigenfunctions by setting the coefficient of the sine function to unity, the normal mode shapes are

$$\phi_1(x) = \sin 0.860\frac{x}{l}$$
$$\phi_2(x) = \sin 3.42\frac{x}{l}$$

As the mode number i becomes large, the natural frequency and normal mode shape can be approximated closely by

$$\omega_i = (i-1)\pi\sqrt{\frac{EI}{\mu l^2}}$$
$$\phi_i(x) = \sin (i-1)\pi\frac{x}{l}$$

The first three normal mode shapes are shown in Fig. 8-17. Note that the displacement of the lumped mass diminishes rapidly with increasing mode number. In the higher modes the lumped mass is nearly stationary.

In the limit, as $m \to 0$, the system approaches that of Fig. 8-7. The frequency equation reduces to

$$\tan \omega_i\sqrt{\frac{\mu l^2}{EA}} = \infty$$

which yields the natural frequencies

$$\omega_i = \frac{(2i-1)\pi}{2}\sqrt{\frac{EA}{\mu l^2}}$$

This result agrees with that of Example 8.3.

or, in terms of the normal coordinates, as

$$T = \frac{1}{2}\int_0^l \mu\left(\sum_{i=1}^{\infty} \phi_i \dot{q}_i\right)\left(\sum_{j=1}^{\infty} \phi_j \dot{q}_j\right) dx + \tfrac{1}{2}m\left(\sum_{i=1}^{\infty} \phi_i(l)\dot{q}_i\right)\left(\sum_{j=1}^{\infty} \phi_j(l)\dot{q}_j\right)$$
$$= \frac{1}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \dot{q}_i\dot{q}_j\left[\int_0^l \mu\phi_i\phi_j\,dx + m\phi_i(l)\phi_j(l)\right]$$

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$$\begin{aligned} q_i(0) &= \frac{1}{M_i}\left[\int_0^l \mu u_0\phi_i\,dx + mu_0(l)\phi_i(l)\right] \\ \dot{q}_i(0) &= \frac{1}{M_i}\left[\int_0^l \mu\dot{u}_0\phi_i\,dx + m\dot{u}_0(l)\phi_i(l)\right] \end{aligned} \tag{8.90}$$

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The solutions $B_1 = 0$ or $\omega = 0$ are trivial, since they lead to $U(x) = 0$. Setting the term in the brackets to zero

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This result agrees with that of Example 8.3

or, in terms of the normal coordinates, as

$$T = \frac{1}{2}\int_0^l \mu\left(\sum_{i=1}^{\infty}\phi_i\dot{q}_i\right)\left(\sum_{j=1}^{\infty}\phi_j\dot{q}_j\right)dx + \tfrac{1}{2}m\left(\sum_{i=1}^{\infty}\phi_i(l)\dot{q}_i\right)\left(\sum_{j=1}^{\infty}\phi_j(l)\dot{q}_j\right)$$

$$= \frac{1}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\dot{q}_i\dot{q}_j\left[\int_0^l \mu\phi_i\phi_j\,dx + m\phi_i(l)\phi_j(l)\right]$$

In view of the first of the orthogonality relations, Eq. (8.88), we can simplify the results to

$$T = \frac{1}{2}\sum_{i=1}^{\infty} M_i\dot{q}_i^2$$

in which the generalized masses are defined by

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Making use of the first of the orthogonality relations, Eqs. (8.88), we can write the initial conditions on the normal coordinates as

$$\begin{aligned} q_i(0) &= \frac{1}{M_i}\left[\int_0^l \mu u_0\phi_i\,dx + mu_0(l)\phi_i(l)\right] \\ \dot{q}_i(0) &= \frac{1}{M_i}\left[\int_0^l \mu\dot{u}_0\phi_i\,dx + m\dot{u}_0(l)\phi_i(l)\right] \end{aligned} \tag{8.90}$$

This result is obtained by following the general procedure which led to Eqs. (8.72). As an alternate, the initial conditions on the normal coordinates are given by Eqs. (8.73).

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The solutions $B_1 = 0$ or $\omega = 0$ are trivial, since they lead to $U(x) = 0$. Setting the term in the brackets to zero

$$\omega\sqrt{\frac{\mu l^2}{EA}}\tan\omega\sqrt{\frac{\mu l^2}{EA}} = \frac{\mu l}{m}$$

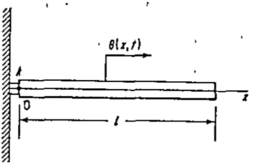


Fig. 8-18

Let us use the principle of virtual displacements to determine the form of the orthogonality relations. If the rod is displaced in the ith mode, the torque in the rod is given by $GJ\phi_i' q_i$. The torque on the spring is given by $k\phi_i(0)q_i$. In a virtual displacement of the jth mode, the relative rotation of the ends of an element of length dx is given by $\phi_j'\,\delta q_j\,dx$. Then the change in the strain energy of the element can be written as $(GJ\phi_i' q_i)(\phi_j'\,\delta q_j)\,dx$. We can write the change in strain energy of the system resulting from the virtual displacement as

$$\begin{aligned}\delta U &= \int_0^l (GJ\phi_i' q_i)(\phi_j'\,\delta q_j)\,dx + [k\phi_i(0)q_i][\phi_j(0)\,\delta q_j] \\ &= q_i\,\delta q_j\left[\int_0^l GJ\phi_i'\phi_j'\,dx + k\phi_i(0)\phi_j(0)\right]\end{aligned}$$

The work done by the elastic forces of the system is the negative of the change in strain energy, as given by $\delta W_{ei} = -\delta U$. In order for the ith and jth modes to be orthogonal, the expression in brackets must be zero. Since the addition of the lumped stiffness can have no effect on the expression for δW_{in}, the work done by the inertia forces in the virtual displacement, we expect the other orthogonality relation to be the same as before.

The ith and jth eigenvalues and normal mode shapes must satisfy the first of the end conditions, Eqs. (8.92), expressed as

$$\begin{aligned}GJ\phi_i'(0) - k\phi_i(0) &= 0 \\ GJ\phi_j'(0) - k\phi_j(0) &= 0\end{aligned}$$

Multiply these equations by $\phi_j(0)$ and $\phi_i(0)$ respectively and add them to the first and second of the following equations.

$$\begin{aligned}GJ\phi_i'\phi_j\Big|_0^l - \int_0^l GJ\phi_i'\phi_j'\,dx &= -\omega_i^2\int_0^l I\phi_i\phi_j\,dx \\ GJ\phi_j'\phi_i\Big|_0^l - \int_0^l GJ\phi_i'\phi_j'\,dx &= -\omega_j^2\int_0^l I\phi_i\phi_j\,dx\end{aligned} \qquad (8.93)$$

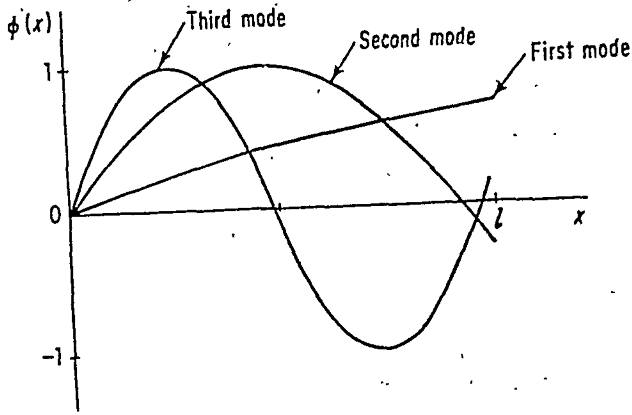


Fig. 8-17 *Normal mode shapes for the system of Fig. 8-16 with $\mu l/m = 1$.*

The limit as $\mu l \to 0$ is also of interest. Since the rod is now a massless spring, the system is a single degree of freedom system. All of the natural frequencies but one become infinite. This one natural frequency is given by

$$\omega = \sqrt{\frac{EA}{ml}}$$

in which EA/l is the stiffness of the spring.

(*b*) *Torsional Vibrations of a Rod with a Lumped Torsional Stiffness*

Consider the free torsional vibrations of the elastic rod of Fig. 8-18. The left-hand end of the rod is attached to a torsional spring of stiffness k. Matching the torque in the spring to that in the rod at $x = 0$, we can write the end conditions as

$$\begin{aligned} T(0,t) &= GJ\theta'(x,t)|_{x=0} = k\theta(0,t) \\ T(l,t) &= GJ\theta'(x,t)|_{x=l} = 0 \end{aligned} \tag{8.91}$$

The equation of motion for the rod is given by Eq. (8.45). Using the method of separation of variables, the functions $\Theta(x)$ and $T(t)$ of Eq. (8.46) must satisfy Eqs. (8.47). Substitution of Eq. (8.46) into Eqs. (8-91) leads to the end conditions on $\Theta(x)$

$$\begin{aligned} GJ\Theta'(0) - k\theta(0) &= 0 \\ GJ\Theta'(l) &= 0 \end{aligned} \tag{8.92}$$

The eigenfunctions $\Theta(x)$ and the associated eigenvalues represent the solution of the first of Eqs. (8.47) consistent with the end conditions, Eqs. (8.92). Normalization of the eigenfunctions $\Theta(x)$ leads to the normal mode shapes $\phi(x)$ for the free torsional vibrations of the rod.

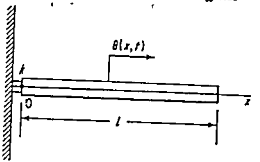


Fig. 8-18

Let us use the principle of virtual displacements to determine the form of the orthogonality relations. If the rod is displaced in the ith mode, the torque in the rod is given by $GJ\phi_i' q_i$. The torque on the spring is given by $k\phi_i(0)q_i$. In a virtual displacement of the jth mode, the relative rotation of the ends of an element of length dx is given by $\phi_j' \delta q_j\, dx$. Then the change in the strain energy of the element can be written as $(GJ\phi_i' q_i)(\phi_j' \delta q_j)\, dx$. We can write the change in strain energy of the system resulting from the virtual displacement as

$$\begin{aligned}\delta U &= \int_0^l (GJ\phi_i' q_i)(\phi_j' \delta q_j)\, dx + [k\phi_i(0)q_i][\phi_j(0)\, \delta q_j] \\ &= q_i\, \delta q_j \left[\int_0^l GJ\phi_i'\phi_j'\, dx + k\phi_i(0)\phi_j(0)\right]\end{aligned}$$

The work done by the elastic forces of the system is the negative of the change in strain energy, as given by $\delta W_{ei} = -\delta U$. In order for the ith and jth modes to be orthogonal, the expression in brackets must be zero. Since the addition of the lumped stiffness can have no effect on the expression for δW_{in}, the work done by the inertia forces in the virtual displacement, we expect the other orthogonality relation to be the same as before.

The ith and jth eigenvalues and normal mode shapes must satisfy the first of the end conditions, Eqs (8 92), expressed as

$$\begin{aligned}GJ\phi_i'(0) - k\phi_i(0) &= 0 \\ GJ\phi_j'(0) - k\phi_j(0) &= 0\end{aligned}$$

Multiply these equations by $\phi_j(0)$ and $\phi_i(0)$ respectively and add them to the first and second of the following equations.

$$\begin{aligned}GJ\phi_i'\phi_j\Big|_0^l - \int_0^l GJ\phi_i'\phi_j'\, dx &= -\omega_i^2 \int_0^l I\phi_i\phi_j\, dx \\ GJ\phi_j'\phi_i\Big|_0^l - \int_0^l GJ\phi_i'\phi_j'\, dx &= -\omega_j^2 \int_0^l I\phi_i\phi_j\, dx\end{aligned} \qquad (8\ 93)$$

These equations correspond to Eqs. (8.58) for longitudinal vibrations. Then

$$\begin{aligned} GJ\phi_i'(l)\phi_j(l) - \int_0^l GJ\phi_i'\phi_j'\,dx - k\phi_i(0)\phi_j(0) &= -\omega_i^2 \int_0^l I\phi_i\phi_j\,dx \\ GJ\phi_j'(l)\phi_i(l) - \int_0^l GJ\phi_i'\phi_j'\,dx - k\phi_i(0)\phi_j(0) &= -\omega_j^2 \int_0^l I\phi_i\phi_j\,dx \end{aligned} \tag{8.94}$$

Referring to the second of the end conditions, Eqs. (8.92), we can write $GJ\phi_i'(x)|_{x=l} = GJ\phi_j'(x)|_{x=l} = 0$. As a result, the integrated terms on the left-hand side of Eqs. (8.94) are zero. If the eigenvalues are distinct and $i \neq j$, we can write the orthogonality relations as

$$\begin{aligned} \int_0^l I\phi_i\phi_j\,dx &= 0 \\ \int_0^l GJ\phi_i'\phi_j'\,dx + k\phi_i(0)\phi_j(0) &= 0 \end{aligned} \tag{8.95}$$

Comparing this result with Eqs. (8.74), the first relation is unaffected. The addition of the lumped stiffness adds a term to the second relation.

Let us write the equations of motion in the normal coordinates by making use of the equation of motion of the rod, given by Eq. (8.10). Substitution of the general solution in the normal coordinates into Eq. (8.10) leads to

$$\sum_{i=1}^{\infty} [(GJ\phi_i')'q_i - I\phi_i\ddot{q}_i] = 0$$

If we multiply this equation by ϕ_j and integrate over the length of the rod, we arrive at

$$\sum_{i=1}^{\infty} \left[q_i \int_0^l (GJ\phi_i')'\phi_j\,dx - \ddot{q}_i \int_0^l I\phi_i\phi_j\,dx\right] = 0$$

Integrating by parts, we can write

$$\sum_{i=1}^{\infty} \left[q_i\left(GJ\phi_i'\phi_j\Big|_0^l - \int_0^l GJ\phi_i'\phi_j'\,dx\right) - \ddot{q}_i \int_0^l I\phi_i\phi_j\,dx\right] = 0$$

Substitution of the general solution in the normal coordinates into the end conditions, Eqs. (8.91) leads to

$$\sum_{i=1}^{\infty} [GJ\phi_i'(0)q_i - k\phi_i(0)q_i] = 0$$

$$\sum_{i=1}^{\infty} GJ\phi_i'(l)q_i = 0$$

If we multiply these equations by $\phi_j(0)$ and $\phi_j(l)$ respectively and use the results to simplify the previous equation, we can write

$$\sum_{i=1}^{\infty} \left\{-q_i\left[\int_0^l GJ\phi_i'\phi_j'\,dx + k\phi_i(0)\phi_j(0)\right] - \ddot{q}_i \int_0^l I\phi_i\phi_j\,dx\right\} = 0$$

As a result of the orthogonality relations, Eqs. (8.95), we can simplify the equation to

$$-M_j\ddot{q}_j - K_j q_j = 0 \qquad j = 1, 2, 3, \ldots$$

in which the generalized mass and stiffness are defined by

$$\begin{aligned} M_j &= \int_0^l I\phi_i^2\, dx \\ K_j &= \int_0^l GJ\phi_i'^2\, dx + k\phi_i^2(0) \end{aligned} \tag{8.96}$$

The generalized mass is unchanged from that given by Eq. (8.75). However the generalized stiffness, given before by Eq. (8.75), has been modified by a term involving the lumped stiffness.

Applying the second of the orthogonality relations, Eqs. (8.95), we can write the initial conditions on the normal coordinates as

$$\begin{aligned} q_i(0) &= \frac{1}{K_i}\left[\int_0^l GJ\theta_0'\phi_i'\, dx + k\theta_0(l)\phi_i(l)\right] \\ \dot{q}_i(0) &= \frac{1}{K_i}\left[\int_0^l GJ\dot{\theta}_0'\phi_i'\, dx + k\dot{\theta}_0(l)\phi_i(l)\right] \end{aligned} \tag{8.97}$$

Instead, we can express the initial conditions in the form given by Eqs. (8.76).

EXAMPLE *8.9*

Let us assume that the rod of Fig. 8-18 is uniform. Substitution of the end conditions, Eqs (8.92), into the general solution for the eigenfunctions $\Theta(x)$, given by Eq. (8.50), leads to

$$\omega\sqrt{\frac{I}{GJ}}B_1 - \frac{k}{GJ}B_2 = 0$$

$$\omega\sqrt{\frac{I}{GJ}}\cos\omega\sqrt{\frac{I}{GJ}}\,lB_1 - \omega\sqrt{\frac{I}{GJ}}\sin\omega\sqrt{\frac{I}{GJ}}\,lB_2 = 0$$

Setting the determinant of the coefficients to zero, the resulting frequency equation is

$$\omega\sqrt{\frac{Il^2}{GJ}}\tan\omega\sqrt{\frac{Il^2}{GJ}} = \frac{kl}{GJ}$$

The trivial solution $\omega = 0$ has been discarded. For the special case in which $kl/GJ = 1$, solution of the frequency equation results in

$$\omega_1 = 0.860\sqrt{\frac{GJ}{Il^2}}$$

$$\omega_2 = 3.42\sqrt{\frac{GJ}{Il^2}}$$

$$\vdots$$

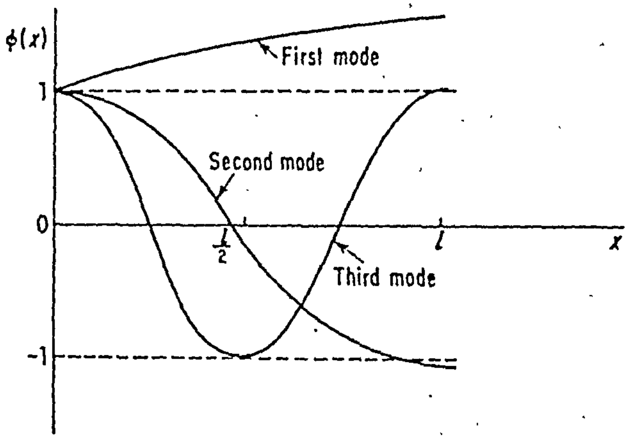


Fig. 8-19 *Normal mode shapes for the system of Fig. 8-18 with $kl/GJ = 1$.*

Normalizing the eigenfunctions by setting the coefficients of the cosine function to unity, we can write the normal mode shapes as

$$\phi_1(x) = 1.163 \sin 0.860 \frac{x}{l} + \cos 0.860 \frac{x}{l}$$

$$\phi_2(x) = 0.292 \sin 3.42 \frac{x}{l} + \cos 3.42 \frac{x}{l}$$

$$\vdots$$

For a large mode number i, the natural frequency and normal mode shape are given very closely by

$$\omega_i = (i - 1)\pi \sqrt{\frac{GJ}{Jl^2}}$$

$$\phi_i(x) = \cos (i - 1)\pi \frac{x}{l}$$

The first three normal mode shapes are shown in Fig. 8-19. Note that the displacement of the tip of the rod approaches ± 1 rapidly with increasing mode number. In the higher modes, the mode shapes are nearly identical with those of the free-free rod considered in Example 8.4.

As we soften the torsional spring with $k \to 0$, the system approaches a free-free rod. The frequency equation reduces to

$$\tan \omega \sqrt{\frac{Jl^2}{GJ}} = 0$$

which yields the natural frequencies

$$\omega_i = (i-1)\pi\sqrt{\frac{Jl^2}{GJ}} \qquad i = 1, 2, 3, \ldots$$

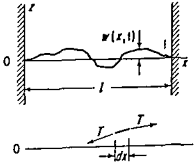
For $i = 1$, the natural frequency is zero as it should be for a rigid-body mode. This result compares with that of Example 8.4.

Problems

8-1 A flexible string of length l and mass per unit length μ is stretched under a large tension T. Show that the linear equation of motion for a free transverse vibration of small amplitude in $w(x,t)$ is given by

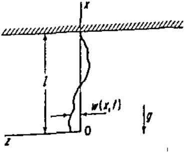
$$T\frac{\partial^2 w}{\partial x^2} - \mu\frac{\partial^2 w}{\partial t^2} = 0$$

Consider the transverse motion of the element of infinitesimal length dx.



Prob. 8-1

8-2 The string of Prob. 8-1 is supported at one end and hangs in a gravitational force field as shown. Using the principle of virtual displacements,



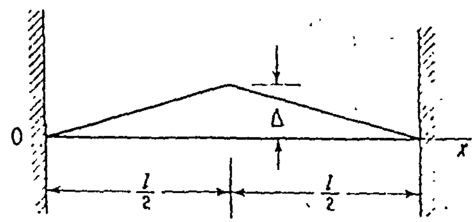
Prob. 8-2

show that the equation of motion for free transverse vibrations of a uniform string of small amplitude is given by $\mu g \frac{\partial}{\partial x}\left(x \frac{\partial w}{\partial x}\right) - \mu \frac{\partial^2 w}{\partial t^2} = 0$. Give the appropriate end conditions.

8-3 Use the principle of virtual displacements to verify the equation of motion, Eq. (8.17), for the flexural vibrations of a beam.

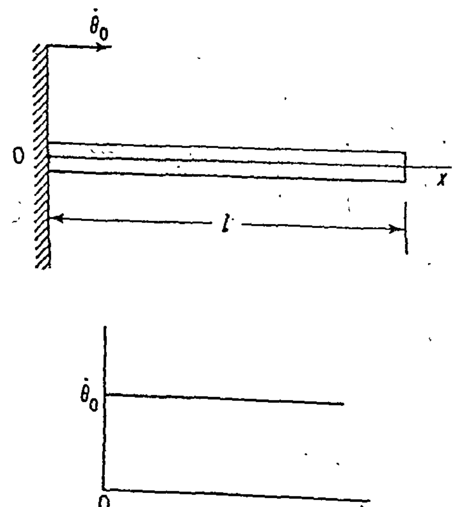
8-4 The equation of motion derived in Prob. 8-1 has the form of the wave equation. If a uniform string is initially displaced as shown and is then released, give the wave solution for the transverse displacement $x(w,t)$ in the form

$$w(x,t) = w_1(x - ct) + w_2(x + ct)$$



Prob. 8-4

8-5 The base of a uniform rod is given a stepwise axial angular velocity $\dot{\theta}_0$ at $t = 0$ as shown. Give the wave solution for the rotation $\theta(x,t)$ of the



Prob. 8-5

elements of the rod relative to the base in the form

$$\theta(x,t) = \theta_1(x - ct) + \theta_2(x + ct)$$

Obtain the wave solution for the torque in the rod as

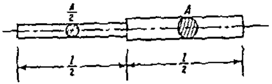
$$T(x,t) = T_1(x - ct) + T_2(x + ct)$$

8-6 Determine the natural frequencies and normal mode shapes for the free longitudinal vibrations of a uniform rod having free ends.

8-7 A straight uniform rod is built in at one end and free at the other. Obtain the natural frequencies and normal mode shapes for free torsional vibrations of the rod.

8-8 Assume that the string of Prob. 8-1 is uniform. Determine the natural frequencies and normal mode shapes for the free transverse vibration of the string.

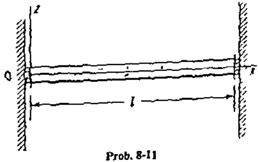
8-9 A rod is made up of two uniform sections of area $A/2$ and A as shown. Assuming the ends to be free, obtain the natural frequencies and normal mode shapes for free longitudinal vibrations. The mass density of the material is given by ρ. Note that the general solution can be written for each section. At the midpoint of the rod, the displacements and forces in the two sections must be compatible.



Prob. 8-9

8-10 Determine the natural frequencies and normal mode shapes for the free flexural vibrations of a uniform cantilever beam.

8-11 The ends of a uniform beam are free to translate in the transverse direction but are constrained against rotation. Obtain the natural frequencies and normal mode shapes for free flexural vibrations of the beam.



Prob. 8-11

the relative elastic displacement, given by $u_E(x,t)$. We can define the flexibility influence coefficient $C(x,\xi)$ as the relative elastic displacement u_E at x resulting from static application of a unit force at ξ. The integral equation of motion, given earlier by Eq. (9.1), becomes

$$u_E(x,t) = -\int_0^l C(x,\xi)\mu(\xi)[\ddot{u}_R(t) + \ddot{u}_E(\xi,t)]\,d\xi \tag{9.6}$$

Since the sum of the inertia forces acting on the rod must be zero in a free vibration, we can write

$$\int_0^l \mu(x)[\ddot{u}_R(t) + \ddot{u}_E(x,t)]\,dx = 0 \tag{9.7}$$

To determine the free vibration, Eqs. (9.6) and (9.7) must be solved simultaneously. If the frame of reference is fixed to the center of mass, the acceleration of the center of mass must be zero, leading to $\ddot{u}_R = 0$. For this important special case, Eqs. (9.6) and (9.7) reduce to the single integral equation

$$u_E(x,t) = -\int_0^l C(x,\xi)\mu(\xi)\ddot{u}_E(\xi,t)\,d\xi \tag{9.8}$$

Evidently it will be an advantage to fix the moving frame of reference to the center of mass.

Although it is possible to define a stiffness influence function $k(x,\xi)$, we will find that it is not used in practical calculations. Thus we will not consider it further.

EXAMPLE 9.1

Consider the uniform rod fixed at one end and free at the other as shown in Fig. 9-2. Let us apply a unit force at ξ and determine the longitudinal displacement $u(x)$ for any x and ξ. We can write the equation for the equilibrium of the rod as

$$\begin{aligned} EAu' &= 1 \qquad \text{for } 0 \le x \le \xi \\ &= 0 \qquad \text{for } \xi \le x \le l \end{aligned}$$

From the definition, the solution for $u(x)$ is just the flexibility influence function $C(x,\xi)$. Solving the equation of equilibrium, we can write

$$\begin{aligned} C(x,\xi) &= \frac{x}{EA} \qquad \text{for } 0 \le x \le \xi \\ &= \frac{\xi}{EA} \qquad \text{for } \xi \le x \le l \end{aligned}$$

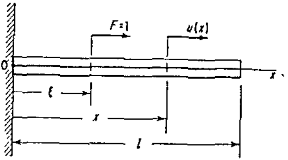


Fig. 9-2 *Flexibility influence function for a cantilever rod.*

We have used the conditions that $u(0) = 0$ and that $u(x)$ is continuous at $x = \xi$. The equation in the eigenvalues and eigenfunctions, the first of Eqs. (9.5), is

$$U(x) = \mu\omega^2\left[\int_0^x \frac{\xi}{EA} U(\xi)\, d\xi + \frac{x}{EA}\int_x^l U(\xi)\, d\xi\right]$$

The two integrals result from the fact that the expression for the influence function changes as the variable of integration ξ passes through the value x.

EXAMPLE 9.2

Let us determine the flexibility influence function for the uniform free-free rod shown in Fig. 9-3. Static application of the unit force at ξ results in a balancing distributed inertia force $f_{in} = -\frac{1}{l}$ as shown. Considering the axial force in the rod, the equation of equilibrium for the rod is given by

$$\begin{aligned} EAu' &= \frac{x}{l} && \text{for } 0 \le x \le \xi \\ &= \frac{x}{l} - 1 && \text{for } \xi \le x \le l \end{aligned}$$

If the moving frame of reference is fixed to the left-hand end of the rod, the conditions on the solution are that $u(0) = 0$ and that $u(x)$ be continuous at $x = \xi$. We can write the solution for the influence function as

$$\begin{aligned} C(x,\xi) &= \frac{l}{EA}\,\frac{1}{2}\left(\frac{x}{l}\right)^2 && \text{for } 0 \le x \le \xi \\ &= \frac{l}{EA}\left[\frac{1}{2}\left(\frac{x}{l}\right)^2 - \frac{x}{l} + \frac{\xi}{l}\right] && \text{for } \xi \le x \le l \end{aligned}$$

The motion of free vibrations is represented by

$$\begin{aligned} u_R(t) &= U_R e^{i\omega t} \\ u_E(x,t) &= U_E(x) e^{i\omega t} \end{aligned}$$

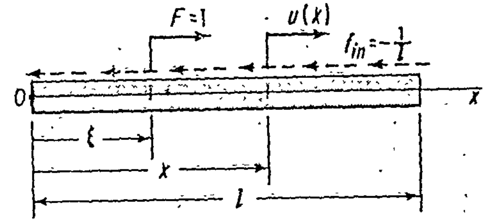


Fig. 9-3 *Flexibility influence function for a free-free rod.*

in which $u_R(t)$ is the displacement of the left-hand end of the rod. From Eqs. (9.6) and (9.7), we can write

$$U_E(x) = \frac{\mu l \omega^2}{EA}\left[\int_0^x \left[\frac{1}{2}\left(\frac{x}{l}\right)^2 - \frac{x}{l} + \frac{\xi}{l}\right][U_R + U_E(\xi)]\,d\xi + \frac{1}{2}\left(\frac{x}{l}\right)^2 \int_x^l [U_R + U_E(\xi)]\,d\xi\right]$$

$$\int_0^l [U_R + U_E(x)]\,dx = 0$$

Simultaneous solution of this pair of equations will yield the eigenvalues ω^2 of the rod. Corresponding to each eigenvalue will be an eigenfunction, given by $U_R + U_E(x)$.

Suppose the moving frame of reference is attached to the center of mass of the rod. Then the solution for $u(x)$ must satisfy the conditions that $\int_0^l \mu u(x)\,dx = 0$ and that $u(x)$ be continuous at $x = \xi$. The solution for the influence function is given by

$$C(x,\xi) = \frac{l}{EA}\left[\frac{1}{2}\left(\frac{x}{l}\right)^2 + \frac{1}{3} - \frac{\xi}{l} + \frac{1}{2}\left(\frac{\xi}{l}\right)^2\right] \quad \text{for } 0 \le x \le \xi$$

$$= \frac{l}{EA}\left[\frac{1}{2}\left(\frac{x}{l}\right)^2 - \frac{x}{l} + \frac{1}{3} + \frac{1}{2}\left(\frac{\xi}{l}\right)^2\right] \quad \text{for } \xi \le x \le l$$

For a free vibration

$$u_E(x,t) = U_E(x)e^{i\omega t}$$

From Eq. (9.8), the equation in the eigenvalues and eigenfunctions is given by

$$U_E(x) = \frac{\mu l \omega^2}{EA}\left[\int_0^x \left[\frac{1}{2}\left(\frac{x}{l}\right)^2 - \frac{x}{l} + \frac{1}{3} + \frac{1}{2}\left(\frac{\xi}{l}\right)^2\right] U_E(\xi)\,d\xi + \int_x^l \left[\frac{1}{2}\left(\frac{x}{l}\right)^2 + \frac{1}{3} - \frac{\xi}{l} + \frac{1}{2}\left(\frac{\xi}{l}\right)^2\right] U_E(\xi)\,d\xi\right]$$

An eigenfunction $U_E(x)$ corresponds to each of the eigenvalues ω^2.

9.2 The Method of Iteration; Stodola's Method

It will be convenient for us to begin by considering the integral equation of motion. For the longitudinal vibrations of an elastic rod, the eigenvalues and the eigenfunctions must satisfy the first of Eqs. (9.5). We can write

$$\frac{1}{\omega^2} U(x) = \int_0^l C(x,\xi)\mu(\xi)U(\xi)\, d\xi \tag{9.9}$$

Let us choose a trial eigenfunction $U^{(0)}(x)$, arbitrarily normalizing the function by setting $U^{(0)}(l) = 1$. Substitution of the trial eigenfunction into the right-hand side of Eq. (9.9) leads to

$$N^{(1)}(x) = \int_0^l C(x,\xi)\mu(\xi)U^{(0)}(\xi)\, d\xi \tag{9.10}$$

If we divide $N^{(1)}(x)$ by $N^{(1)}(l)$, the resulting function will have the value unity at $x = l$. Comparing this result with the left-hand side of Eq. (9.9), we can write

$$\begin{aligned} U^{(1)}(x) &= \frac{N^{(1)}(x)}{N^{(1)}(l)} \\ \frac{1}{\omega^2_{(1)}} &= N^{(1)}(l) \end{aligned} \tag{9.11}$$

in which $U^{(1)}(x)$ is an improved trial eigenfunction and $\omega^2_{(1)}$ is a first estimate of the associated eigenvalue. If $U^{(0)}(x)$ was an exact eigenfunction of the system, the resulting function $U^{(1)}(x)$ will be unchanged and $\omega^2_{(1)}$ will be the exact eigenvalue. Otherwise $U^{(1)}(x)$ will differ from $U^{(0)}(x)$. We can continue the iterative process by substituting the improved trial eigenfunction into the right-hand side of Eq. (9.9), leading to

$$N^{(2)}(x) = \int_0^l C(x,\xi)\mu(\xi)U^{(1)}(\xi)\, d\xi \tag{9.12}$$

and to

$$\begin{aligned} U^{(2)}(x) &= \frac{N^{(2)}(x)}{N^{(2)}(l)} \\ \frac{1}{\omega^2_{(2)}} &= N^{(2)}(l) \end{aligned} \tag{9.13}$$

The quantities $U^{(2)}(x)$ and $\omega^2_{(2)}$ represent a further improved trial eigenfunction and a second estimate of the eigenvalue respectively. We can show that the results converge on the fundamental eigenfunction $U_1(x)$ and eigenvalue ω_1^2.

Any possible configuration of the rod can be described by a linear combination of the exact eigenfunctions. We can write the original trial eigenfunction as

$$U^{(0)}(x) = \sum_{i=1}^{\infty} U_i(x)c_i \tag{9.14}$$

in which the coefficients c_i represent the amplitudes of the eigenfunctions. Substitution of Eq. (9.14) into the right-hand side of Eq. (9.9) leads to

$$N^{(1)}(x) = \sum_{i=1}^{\infty} c_i \int_0^l C(x,\xi)\mu(\xi)U_i(\xi)\,d\xi \tag{9.15}$$

The ith eigenfunction and eigenvalue must satisfy Eq. (9.9), written as

$$\frac{1}{\omega_i^2} U_i(x) = \int_0^l C(x,\xi)\mu(\xi)U_i(\xi)\,d\xi \tag{9.16}$$

Combination of Eqs. (9.15) and (9.16) results in

$$N^{(1)}(x) = \sum_{i=1}^{\infty} c_i \frac{1}{\omega_i^2} U_i(x) \tag{9.17}$$

The improved trial eigenfunction is given by

$$U^{(1)}(x) = \frac{1}{N^{(1)}(l)} \sum_{i=1}^{\infty} c_i \frac{1}{\omega_i^2} U_i(x) \tag{9.18}$$

Continuing the procedure through r iterations, we can write

$$U^{(r)}(x) = \frac{1}{N^{(1)}(l)\cdots N^{(r)}(l)} \sum_{i=1}^{\infty} c_i\left(\frac{1}{\omega_i^2}\right)^r U_i(x) \tag{9.19}$$

Since $\omega_1 < \omega_2 < \cdots$, it is evident that the first term in the series will become dominant after several iterations. If $c_1 \neq 0$, we can approximate Eq. (9.19) by

$$U^{(r)}(x) = \frac{1}{N^{(1)}(l)\cdots N^{(r)}(l)} c_1 \frac{1}{\omega_1^{2r}} U_1(x) \tag{9.20}$$

Thus the trial eigenfunction converges on the fundamental eigenfunction $U_1(x)$ of the rod. If we have chosen our original trial eigenfunction to represent the exact fundamental eigenfunction closely, the coefficient c_1 will be relatively large and the convergence will be hastened. The eigenfunction will be normalized with $U_1(l) = 1$. We could of course have normalized the function at any other convenient point on the rod. The fundamental eigenvalue is approximated more and more closely by

$$\frac{1}{\omega_1^2} = N^{(r)}(l) \tag{9.21}$$

as r increases.

The basic method just outlined is referred to as Stodola's method. It is not necessary to base the procedure on the integral equation Assume that the rod is vibrating with a simple harmonic motion of frequency ω and of shape $U^{(0)}(x)$, the trial eigenfunction. The distributed inertia loading on the rod is given by $-\mu\omega^2 U^{(0)}(x)$. Let us determine the displacement of the rod resulting from the inertia loading using any of the available methods The resulting shape function, normalized in the same way as was $U^{(0)}(x)$, represents an improved trial eigenfunction $U^{(1)}(x)$. This procedure can be continued until it produces a good estimate of the fundamental eigenfunction $U_1(x)$.

We can extend the iterative procedure to produce the second eigenfunction and eigenvalue by using the property of orthogonality of the normal modes. A trial eigenfunction $U^{(0)}(x)$ will be orthogonal to the fundamental eigenfunction $U_1(x)$ if we require

$$\int_0^l \mu(x) U^{(0)}(x) U_1(x)\, dx = 0 \tag{9.22}$$

If the trial eigenfunction satisfies Eq (9.22), we can expect the iterative procedure to converge on the second eigenfunction and eigenvalue of the rod. In this manner, we can obtain the characteristics of several of the normal modes.

EXAMPLE 9.3

Let us obtain the fundamental eigenvalue and eigenfunction of the uniform rod of Fig. 9-2 using the method of iteration From the results of Example 9.1,

$$\frac{1}{\omega^2} U(x) = \mu\left[\int_0^x \frac{\xi}{EA} U(\xi)\, d\xi + \frac{x}{EA}\int_x^l U(\xi)\, d\xi\right]$$

Suppose we select for the trial eigenfunction

$$U^{(0)}(x) = \frac{x}{l}$$

which is normalized with $U^{(0)}(l) = 1$. If we substitute the trial eigenfunction into the right-hand side of the previous equation and perform the integration, we will arrive at

$$N^{(1)}(x) = \frac{\mu l^2}{EA}\left[\frac{1}{2}\frac{x}{l} - \frac{1}{6}\left(\frac{x}{l}\right)^3\right]$$

with

$$N^{(1)}(l) = \frac{1}{3}\frac{\mu l^2}{EA}$$

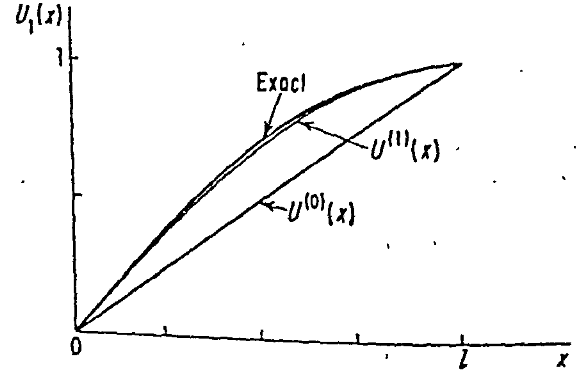


Fig. 9-4

We can write the improved trial eigenfunction as

$$U^{(1)}(x) = \frac{3}{2}\frac{x}{l} - \frac{1}{2}\left(\frac{x}{l}\right)^3$$

and the first estimate of the associated eigenvalue as

$$\omega^2{}_{(1)} = 3\frac{EA}{\mu l^2}$$

Repeating the procedure, we will obtain the result

$$U^{(2)}(x) = \frac{25}{16}\frac{x}{l} - \frac{5}{8}\left(\frac{x}{l}\right)^3 + \frac{1}{16}\left(\frac{x}{l}\right)^5$$

$$\omega^2{}_{(2)} = \frac{5}{2}\frac{EA}{\mu l^2}$$

By comparison, the exact results from Example 8.3 are

$$U_1(x) = \sin\frac{\pi x}{2l}$$

$$\omega_1^2 = \left(\frac{\pi}{2}\right)^2\frac{EA}{\mu l^2} = 2.47\frac{EA}{\mu l^2}$$

The results for the fundamental eigenfunction are shown in Fig. 9-4. After two iterations the trial eigenfunction is so close to the exact result that they cannot be distinguished from each other in the figure. In a third iteration, the result for the eigenvalue will be correct to three significant figures.

9.3 The Method of Collocation

(a) *Applied to the Integral Equation of Motion*

For a continuous system, the integral equation of motion represents an infinite number of ordinary equations of motion in an infinite number of

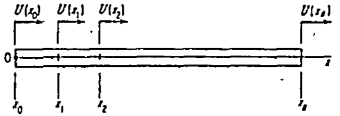


Fig. 9-5 *Selection of stations for the method of collocation.*

coordinates. The method of collocation is an approximate method for writing a finite number of equations of motion in an equal number of coordinates. To illustrate the method, let us consider the longitudinal vibrations of the elastic rod of Fig. 9-5. The stations $x_0, x_1, \ldots, x_n$ represent $n + 1$ points on the rod, not necessarily evenly spaced. According to the method of collocation, we will require the equation of motion to be satisfied at each of the $n + 1$ stations. Between stations, the equation of motion will not in general be satisfied. If at least one of the ends is fixed, we can use the first of Eqs. (9.5) to obtain

$$U(x_i) = \omega^2 \int_0^l C(x_i, \xi)\mu(\xi)U(\xi)\, d\xi \qquad i = 0, 1, \ldots, n \tag{9.23}$$

If both ends are fixed, the two equations with $i = 0, n$ reduce to $0 = 0$, leaving $n - 1$ equations of motion. Similarly if only one end is fixed, we will be left with n equations of motion. In the event that both ends are free, we can write a set of equations similar to Eq. (9.23) by starting with the appropriate equations of motion, either Eqs. (9.6) and (9.7) or Eq. (9.8). If we refer the motion to a frame of reference fixed to the center of mass and start with Eq. (9.8), the resulting equations will have the form of Eqs. (9.23).

In the following, let us assume that the left- and right-hand ends are fixed and free respectively. Since $U(0) = 0$ and $C(0, \xi) = 0$, the first of Eqs. (9.23), that for $n = 0$, reduces to $0 = 0$. In order to solve the remaining n equations, we will need to select a reduced set of n coordinates. Consider the n generalized coordinates p described by

$$U(x) = \sum_{j=1}^{n} \gamma_j(x)p_j \tag{9.24}$$

in which the shape functions γ_j are required to satisfy $\gamma_j(0) = 0$. Substitution of Eq. (9.24) into Eq. (9.23) leads to

$$\sum_{j=1}^{n} p_j\left[\gamma_j(x_i) - \omega^2 \int_0^l C(x_i, \xi)\mu(\xi)\gamma_j(\xi)\, d\xi\right] = 0 \qquad i = 1, 2, \ldots, n \tag{9.25}$$

We can solve this set of equations for the n eigenvalues and eigenvectors using methods outlined for lumped-mass systems in Chaps. 5 and 6. In this case

the term eigenvector refers to a particular set of values for the coordinates p. Substitution of this set of values in Eq. (9.24) yields an approximate mode shape. If the integration involved in Eq. (9.25) is not convenient, it may be necessary to approximate the integral with a numerical integration procedure.

The usual choice of coordinates is that of the displacements at the selected stations, represented by

$$\begin{aligned} U_0 &= U(x_0) \\ U_1 &= U(x_1) \\ &\vdots \\ U_n &= U(x_n) \end{aligned}$$

For the chosen end conditions, $U_0 = 0$. With this choice of coordinates, it will be necessary to replace the integral of Eq. (9.23) by the result of a numerical integration. The val ... egrand at the jth station x_j is given by $C(x_i,x_j)\mu(x_j)U(x_j)$. Evi ... flexibility influence coefficient for the stations x_i,x_j ... unit length of the rod evaluated at the statio ... ons

This set of equations can be solved for the n eigenvectors $\{U\}$ and associated eigenvalues ω^2 using the methods described in Chaps. 5 and 6. In particular, the equations are in the proper form for the iteration procedure which converges on the fundamental mode, that outlined in Sec. 6.1.

EXAMPLE 9.4

Consider the free longitudinal vibrations of the uniform elastic rod shown in Fig. 9-6. Let us select the five evenly spaced stations x_0, x_1, x_2, x_3, x_4 and

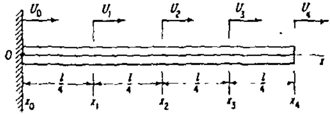


Fig. 9-6

the corresponding coordinates U_0, U_1, U_2, U_3, U_4 as shown. Since $U_0 = 0$, we have approximated the rod as a four degree of freedom system. The flexibility influence coefficients associated with the points x_1, x_2, x_3, x_4 are given by

$$[C] = \frac{l}{4EA}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

For the numerical integration, let us use the trapezoidal rule. Refering to Appendix D, the weighting numbers are given by

$$[w] = \frac{l}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Since the rod is uniform, we can write

$$[\mu] = \mu\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

the term eigenvector refers to a particular set of values for the coordinates p. Substitution of this set of values in Eq. (9.24) yields an approximate mode shape. If the integration involved in Eq. (9.25) is not convenient, it may be necessary to approximate the integral with a numerical integration procedure.

The usual choice of coordinates is that of the displacements at the selected stations, represented by

$$\begin{aligned} U_0 &= U(x_0) \\ U_1 &= U(x_1) \\ &\vdots \\ U_n &= U(x_n) \end{aligned}$$

For the chosen end conditions, $U_0 = 0$. With this choice of coordinates, it will be necessary to replace the integral of Eq. (9.23) by the result of a numerical integration. The value of the integrand at the jth station x_j is given by $C(x_i,x_j)\mu(x_j)U(x_j)$. Evidently $C(x_i,x_j)$ is the flexibility influence coefficient for the stations x_i,x_j. Further, $\mu(x_j)$ is the mass per unit length of the rod evaluated at the station x_j. Let us make the abbreviations

$$\begin{aligned} C(x_i,x_j) &= C_{ij} \\ \mu(x_j) &= \mu_j \end{aligned}$$

and write the value of the integrand as $C_{ij}\mu_jU_j$. The integral is approximated by the sum of the values of the integrand, each multiplied by a weighting number w. We can write

$$\int_0^l C(x_i,\xi)\mu(\xi)U(\xi)\,d\xi = \sum_{j=1}^{n} C_{ij}\mu_j w_j U_j$$

Refer to Appendix D for a discussion of numerical integration and weighting numbers. The weighting numbers are determined by the rule used for the integration. Now we can write Eqs. (9.23) as

$$U_i = \omega^2 \sum_{j=1}^{n} C_{ij}\mu_j w_j U_j \qquad i = 1, 2, \ldots, n \tag{9.26}$$

In matrix form

$$\begin{Bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{Bmatrix} = \omega^2 \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & \cdot \\ \vdots & & & \vdots \\ C_{n1} & \cdot & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & \cdot \\ \vdots & & & \vdots \\ 0 & \cdot & \cdots & \mu_n \end{bmatrix}$$

$$\times \begin{bmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & \cdots & \cdot \\ \vdots & & & \vdots \\ 0 & \cdot & \cdots & w_n \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{Bmatrix} \tag{9.27}$$

or

$$\{U\} = \omega^2[C][\mu][w]\{U\} \tag{9.28}$$

This set of equations can be solved for the n eigenvectors $\{U\}$ and associated eigenvalues ω^2 using the methods described in Chaps. 5 and 6. In particular, the equations are in the proper form for the iteration procedure which converges on the fundamental mode, that outlined in Sec. 6.1.

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Consider the free longitudinal vibrations of the uniform elastic rod shown in Fig. 9-6. Let us select the five evenly spaced stations x_0, x_1, x_2, x_3, x_4 and

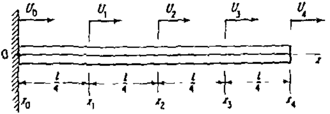

Fig. 9-6

the corresponding coordinates U_0, U_1, U_2, U_3, U_4 as shown. Since $U_0 = 0$, we have approximated the rod as a four degree of freedom system. The flexibility influence coefficients associated with the points x_1, x_2, x_3, x_4 are given by

$$[C] = \frac{l}{4EA}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

For the numerical integration, let us use the trapezoidal rule. Refering to Appendix D, the weighting numbers are given by

$$[w] = \frac{l}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Since the rod is uniform, we can write

$$[\mu] = \mu\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For illustration, let us assume that the left and right-hand ends of the rod are fixed and free respectively. We will need a sufficient number of coordinates to satisfy the n equations of motion, Eq. (9.29), as well as the end conditions, Eqs. (9.30). A possible choice for the $n + 2$ coordinates needed is the set of generalized coordinates p defined by

$$U(x) = \sum_{j=1}^{n+2} \gamma_j(x) p_j \tag{9.31}$$

Substitution of Eq. (9.31) into Eq. (9.29) leads to

$$\sum_{j=1}^{n+2} p_j[(EA\gamma_j')'|_{x=x_i} + \mu\omega^2\gamma_j|_{x=x_i}] = 0 \qquad i = 1, \ldots, n \tag{9.32}$$

For the chosen end conditions, the combination of Eqs. (9.30) and (9.31) results in

$$\begin{aligned} \sum_{j=1}^{n+2} p_j\gamma_j(x_0) &= 0 \\ \sum_{j=1}^{n+2} p_j EA\gamma_j'|_{x=x_n} &= 0 \end{aligned} \tag{9.33}$$

The solution of Eqs. (9.32) and (9.33) yields a set of n eigenvalues and eigenvectors. Substitution of an eigenvector, a particular set of values for the coordinates p, into Eq. (9.31) produces an approximate mode shape. If the chosen functions γ_j individually satisfy the end conditions, as given by $\gamma_j(x_0) = EA\gamma_j'|_{x=x_n} = 0$, we can eliminate Eqs. (9.33). The number of equations will have been reduced to n and we must reduce the number of coordinates p in Eqs. (9.31) and (9.32) to n. The usual practice is to select functions which at least satisfy the displacement conditions, in this case $\gamma_j(x_0) = 0$.

Often we will choose our coordinates to be the displacements at the selected stations, given by $U_0, U_1, \ldots, U_n$. For the special case of the fixed-free rod, the number $n + 1$ of the coordinates is one fewer than the number $n + 2$ of the given equations, Eqs. (9.29) and (9.30). As will be seen in Example 9.5, it is customary to introduce an additional coordinate at a station beyond the free end. For the chosen coordinates it will be necessary to replace the derivatives appearing in the equations by approximate finite difference expressions. Refer to Appendix D for a discussion of the representation of derivatives in terms of a finite number of such coordinates. We will end up with n linear algebraic equations in the coordinates $U_1, U_2, \ldots, U_n$. Solution of the equations yields a set of n eigenvalues ω^2 and corresponding eigenvectors $\{U\}$.

EXAMPLE 9.5

Let us consider again the free longitudinal vibrations of the uniform rod of Fig. 9-6 The spacing of the stations is given by $h = l/4$. Suppose we introduce an additional station x_5 to the right of the free end, letting U_5 represent the displacement at the fictitious station. From Appendix D, we can write the finite-difference expressions for the derivative U'' at the stations x_1, x_2, x_3, x_4 as

$$U''|_{x=x_1} = \frac{16}{l^2}(U_2 - 2U_1 + U_0)$$

$$U''|_{x=x_2} = \frac{16}{l^2}(U_3 - 2U_2 + U_1)$$

$$U''|_{x=x_3} = \frac{16}{l^2}(U_4 - 2U_3 + U_2)$$

$$U''|_{x=x_4} = \frac{16}{l^2}(U_5 - 2U_4 + U_3)$$

Then the equations of motion, Eqs. (9.29), for the stations x_1, x_2, x_3, x_4 are

$$16\frac{EA}{l^2}(U_2 - 2U_1 + U_0) + \mu\omega^2 U_1 = 0$$

$$16\frac{EA}{l^2}(U_3 - 2U_2 + U_1) + \mu\omega^2 U_2 = 0$$

$$16\frac{EA}{l^2}(U_4 - 2U_3 + U_2) + \mu\omega^2 U_3 = 0$$

$$16\frac{EA}{l^2}(U_5 - 2U_4 + U_3) + \mu\omega^2 U_4 = 0$$

The end conditions, Eqs. (9.30), at the stations x_0, x_4 are

$$U(x_0) = U_0 = 0$$

$$U'|_{x=x_4} = \frac{2}{l}(U_5 - U_3) = 0$$

We can use the end conditions to eliminate U_0 and U_5 from the equations of motion. The equations of motion in the matrix form are

$$-\mu\omega^2\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} + 16\frac{EA}{l^2}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Multiplying the first three equations by $l/4$ and the fourth by $l/8$, we can write

$$-\omega^2[m]\{U\} + [k]\{U\} = \{0\}$$

For illustration, let us assume that the left and right-hand ends of the rod are fixed and free respectively. We will need a sufficient number of coordinates to satisfy the n equations of motion, Eq. (9.29), as well as the end conditions, Eqs. (9.30). A possible choice for the $n + 2$ coordinates needed is the set of generalized coordinates p defined by

$$U(x) = \sum_{j=1}^{n+2} \gamma_j(x)p_j \tag{9.31}$$

Substitution of Eq. (9.31) into Eq. (9.29) leads to

$$\sum_{j=1}^{n+2} p_j[(EA\gamma_j')'|_{x=x_i} + \mu\omega^2\gamma_j|_{x=x_i}] = 0 \qquad i = 1, \ldots, n \tag{9.32}$$

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The solution of Eqs. (9.32) and (9.33) yields a set of n eigenvalues and eigenvectors. Substitution of an eigenvector, a particular set of values for the coordinates p, into Eq. (9.31) produces an approximate mode shape. If the chosen functions γ_j individually satisfy the end conditions, as given by $\gamma_j(x_0) = EA\gamma_j'|_{x=x_n} = 0$, we can eliminate Eqs. (9.33). The number of equations will have been reduced to n and we must reduce the number of coordinates p in Eqs. (9.31) and (9.32) to n. The usual practice is to select functions which at least satisfy the displacement conditions, in this case $\gamma_j(x_0) = 0$.

Often we will choose our coordinates to be the displacements at the selected stations, given by $U_0, U_1, \ldots, U_n$. For the special case of the fixed-free rod, the number $n + 1$ of the coordinates is one fewer than the number $n + 2$ of the given equations, Eqs. (9.29) and (9.30). As will be seen in Example 9.5, it is customary to introduce an additional coordinate at a station beyond the free end. For the chosen coordinates it will be necessary to replace the derivatives appearing in the equations by approximate finite difference expressions. Refer to Appendix D for a discussion of the representation of derivatives in terms of a finite number of such coordinates. We will end up with n linear algebraic equations in the coordinates $U_1, U_2, \ldots, U_n$. Solution of the equations yields a set of n eigenvalues ω^2 and corresponding eigenvectors $\{U\}$.

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Let us consider again the free longitudinal vibrations of the uniform rod of Fig. 9-6. The spacing of the stations is given by $h = l/4$. Suppose we introduce an additional station x_5 to the right of the free end, letting U_5 represent the displacement at the fictitious station. From Appendix D, we can write the finite-difference expressions for the derivative U'' at the stations x_1, x_2, x_3, x_4 as

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$$U''|_{x=x_4} = \frac{16}{l^2}(U_5 - 2U_4 + U_3)$$

Then the equations of motion, Eqs. (9.29), for the stations x_1, x_2, x_3, x_4 are

$$16\frac{EA}{l^2}(U_2 - 2U_1 + U_0) + \mu\omega^2 U_1 = 0$$

$$16\frac{EA}{l^2}(U_3 - 2U_2 + U_1) + \mu\omega^2 U_2 = 0$$

$$16\frac{EA}{l^2}(U_4 - 2U_3 + U_2) + \mu\omega^2 U_3 = 0$$

$$16\frac{EA}{l^2}(U_5 - 2U_4 + U_3) + \mu\omega^2 U_4 = 0$$

The end conditions, Eqs. (9.30), at the stations x_0, x_4 are

$$U(x_0) = U_0 = 0$$

$$U'|_{x=x_4} = \frac{2}{l}(U_5 - U_3) = 0$$

We can use the end conditions to eliminate U_0 and U_5 from the equations of motion. The equations of motion in the matrix form are

$$-\mu\omega^2\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} + 16\frac{EA}{l^2}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Multiplying the first three equations by $l/4$ and the fourth by $l/8$, write

$$-\omega^2[m]\{U\} + [k]\{U\} = \{0\}$$

In matrix form, we can write Eqs. (9.40) as

$$\left[\omega^2\begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & \cdot \\ \vdots & & & \vdots \\ M_{n1} & \cdot & \cdots & M_{nn}\end{bmatrix} - \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & \cdot \\ \vdots & & & \vdots \\ K_{n1} & \cdot & \cdots & K_{nn}\end{bmatrix}\right]\begin{Bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \tag{9.42}$$

Using the methods outlined for lumped-mass systems, we can solve this set of equations for the n eigenvalues ω^2 and associated eigenvectors $\{p\}$.

In order to approximate the motion of the rod reasonably well, the number required of the generalized coordinates p is usually much smaller than the number required of the coordinates $U_0, U_1, \ldots, U_n$. Thus Galerkin's method is a more efficient method than the method of collocation in that fewer coordinates and equations are needed. On the other hand, the coefficients involved in the equations are much easier to write when using the method of collocation.

EXAMPLE 9.6

Consider the free longitudinal vibrations of the straight elastic rod shown in Fig. 9-9. The rod is made up of two uniform sections having cross-sectional areas A and $A/2$ as shown and mass per unit length μ and $\mu/2$. Let us describe

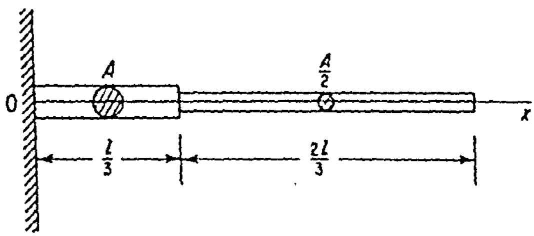

Fig. 9-9

the motion wit[illegible] [illegible]zed [illegible] s whose shape functions are just the normal mode [illegible] e [illegible] ven in Example 8.3. These shape functions satis[illegible] as required. If we elect to use two coordinat[illegible]

From Eqs. (9.41), the generalized masses and stiffnesses are

$$M_{11} = \mu \int_0^{l/3} \sin^2 \frac{\pi x}{2l}\, dx + \frac{\mu}{2} \int_{l/3}^{l} \sin^2 \frac{\pi x}{2l}\, dx$$
$$= 0.2644\mu l$$

$$M_{12} = M_{21} = \mu \int_0^{l/3} \sin \frac{\pi x}{2l} \sin \frac{3\pi x}{2l}\, dx + \frac{\mu}{2} \int_{l/3}^{l} \sin \frac{\pi x}{2l} \sin \frac{3\pi z}{2l}\, dx$$
$$= 0.03446\mu l$$

$$M_{22} = \mu \int_0^{l/3} \sin^2 \frac{3\pi x}{2l}\, dx + \frac{\mu}{2} \int_{l/3}^{l} \sin^2 \frac{3\pi x}{2l}\, dx$$
$$= 0.3333\mu l$$

$$K_{11} = EA\left(\frac{\pi}{2l}\right)^2 \int_0^{l/3} \cos^2 \frac{\pi x}{2l}\, dx + \frac{EA}{2}\left(\frac{\pi}{2l}\right)^2 \int_{l/3}^{l} \cos^2 \frac{3\pi x}{2l}\, dx$$
$$= 0.9928\, \frac{EA}{l}$$

$$K_{12} = K_{21} = EA\, \frac{\pi}{2l}\, \frac{3\pi}{2l} \int_0^{l/3} \cos \frac{\pi x}{2l} \cos \frac{3\pi x}{2l}\, dx + \frac{EA}{2} \cdot \frac{\pi}{2l}\, \frac{3\pi}{2l} \int_{l/3}^{l} \cos \frac{\pi x}{2l} \cos \frac{3\pi x}{2l}\, dx$$
$$= 0.7652\, \frac{EA}{l}$$

$$K_{22} = EA\left(\frac{3\pi}{2l}\right)^2 \int_0^{l/3} \cos^2 \frac{3\pi x}{2l}\, dx + \frac{EA}{2}\left(\frac{3\pi}{2l}\right)^2 \int_{l/3}^{l} \cos^2 \frac{3\pi x}{2l}\, dx$$
$$= 7.402\, \frac{EA}{l}$$

From Eq. (9.42), we can write

$$\left[\mu l\omega^2 \begin{bmatrix} 0.2644 & 0.03446 \\ 0.03446 & 0.3333 \end{bmatrix} - \frac{EA}{l} \begin{bmatrix} 0.9928 & 0.7652 \\ 0.7652 & 7.402 \end{bmatrix}\right] \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Setting the determinant of coefficients to zero, we can write the characteristic equation as

$$\omega^4 - 25.71\, \frac{EA}{\mu l^2}\, \omega^2 + 77.80\left(\frac{EA}{\mu l^2}\right)^2 = 0$$

The eigenvalues are given by

$$\omega_1^2 = 3.50\, \frac{EA}{\mu l^2}$$
$$\omega_2^2 = 22.2\, \frac{EA}{\mu l^2}$$

and the eigenvectors by

$$\{p\}_1 = \begin{Bmatrix} 1 \\ -0.103 \end{Bmatrix}$$
$$\{p\}_2 = \begin{Bmatrix} -0.0003 \\ 1 \end{Bmatrix}$$

In matrix form, we can write Eqs. (9.40) as

$$\left[\omega^2\begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & \cdot \\ \vdots & & & \vdots \\ M_{n1} & \cdot & \cdots & M_{nn} \end{bmatrix} - \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & \cdot \\ \vdots & & & \vdots \\ K_{n1} & \cdot & \cdots & K_{nn} \end{bmatrix}\right]\begin{Bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \tag{9.42}$$

Using the methods outlined for lumped-mass systems, we can solve this set of equations for the n eigenvalues ω^2 and associated eigenvectors $\{p\}$.

In order to approximate the motion of the rod reasonably well, the number required of the generalized coordinates p is usually much smaller than the number required of the coordinates $U_0, U_1, \ldots, U_n$. Thus Galerkin's method is a more efficient method than the method of collocation in that fewer coordinates and equations are needed. On the other hand, the coefficients involved in the equations are much easier to write when using the method of collocation.

EXAMPLE *9.6*

Consider the free longitudinal vibrations of the straight elastic rod shown in Fig. 9-9. The rod is made up of two uniform sections having cross-sectional areas A and $A/2$ as shown and mass per unit length μ and $\mu/2$. Let us describe

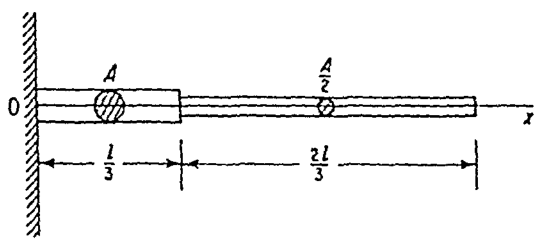

Fig. 9-9

the motion with generalized coordinates whose shape functions are just the normal mode shapes for the uniform rod given in Example 8.3. These shape functions satisfy the end conditions on the rod as required. If we elect to use two coordinates, we can write

$$\gamma_1(x) = \sin\frac{\pi x}{2l}$$

$$\gamma_2(x) = \sin\frac{3\pi x}{2l}$$

From Eqs. (9.41), the generalized masses and stiffnesses are

$$M_{11} = \mu \int_0^{l/3} \sin^2 \frac{\pi x}{2l}\, dx + \frac{\mu}{2}\int_{l/3}^{l} \sin^2 \frac{\pi x}{2l}\, dx$$
$$= 0.2644\mu l$$
$$M_{12} = M_{21} = \mu \int_0^{l/3} \sin \frac{\pi x}{2l} \sin \frac{3\pi x}{2l}\, dx + \frac{\mu}{2}\int_{l/3}^{l} \sin \frac{\pi x}{2l} \sin \frac{3\pi x}{2l}\, dx$$
$$= 0.03446\mu l$$
$$M_{22} = \mu \int_0^{l/3} \sin^2 \frac{3\pi x}{2l}\, dx + \frac{\mu}{2}\int_{l/3}^{l} \sin^2 \frac{3\pi x}{2l}\, dx$$
$$= 0.3333\mu l$$
$$K_{11} = EA\left(\frac{\pi}{2l}\right)^2 \int_0^{l/3} \cos^2 \frac{\pi x}{2l}\, dx + \frac{EA}{2}\left(\frac{\pi}{2l}\right)^2 \int_{l/3}^{l} \cos^2 \frac{3\pi x}{2l}\, dx$$
$$= 0.9928\, \frac{EA}{l}$$
$$K_{12} = K_{21} = EA\, \frac{\pi}{2l}\, \frac{3\pi}{2l} \int_0^{l/3} \cos \frac{\pi x}{2l} \cos \frac{3\pi x}{2l}\, dx + \frac{EA}{2}\, \frac{\pi}{2l}\, \frac{3\pi}{2l} \int_{l/3}^{l} \cos \frac{\pi x}{2l} \cos \frac{3\pi x}{2l}\, dx$$
$$= 0.7652\, \frac{EA}{l}$$
$$K_{22} = EA\left(\frac{3\pi}{2l}\right)^2 \int_0^{l/3} \cos^2 \frac{3\pi x}{2l}\, dx + \frac{EA}{2}\left(\frac{3\pi}{2l}\right)^2 \int_{l/3}^{l} \cos^2 \frac{3\pi x}{2l}\, dx$$
$$= 7.402\, \frac{EA}{l}$$

From Eq. (9.42), we can write

$$\left[\mu l\omega^2 \begin{bmatrix} 0.2644 & 0.03446 \\ 0.03446 & 0.3333 \end{bmatrix} - \frac{EA}{l}\begin{bmatrix} 0.9928 & 0.7652 \\ 0.7652 & 7.402 \end{bmatrix}\right]\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Setting the determinant of coefficients to zero, we can write the characteristic equation as

$$\omega^4 - 25.71\, \frac{EA}{\mu l^2}\,\omega^2 + 77.80\left(\frac{EA}{\mu l^2}\right)^2 = 0$$

The eigenvalues are given by

$$\omega_1^2 = 3.50\, \frac{EA}{\mu l^2}$$
$$\omega_2^2 = 22.2\, \frac{EA}{\mu l^2}$$

and the eigenvectors by

$$\{p\}_1 = \begin{Bmatrix} 1 \\ -0.103 \end{Bmatrix}$$
$$\{p\}_2 = \begin{Bmatrix} -0.0003 \\ 1 \end{Bmatrix}$$

The approximate fundamental mode shape is

$$\phi_1(x) = \sin\frac{\pi x}{2l} - 0.103\sin\frac{3\pi x}{2l}$$

For the exact solution it can be shown that the characteristic equation is

$$\cos\omega\sqrt{\frac{\mu l^2}{EA}} + \tfrac{1}{3}\cos\frac{\omega}{3}\sqrt{\frac{\mu l^2}{EA}} = 0$$

Then the exact fundamental natural frequency is $\omega_1 = 1.843\sqrt{\frac{EA}{\mu l^2}}$. Our estimate of $\omega_1 = 1.87\sqrt{\frac{EA}{\mu l^2}}$ is about 1.5 per cent too high.

(*b*) *A Modification of the method*

It is often inconvenient to generate functions $\gamma(x)$ which will satisfy all of the end conditions. This is particularly true for the end conditions involving forces. Let us examine a modification of Galerkin's method which will permit us to use functions which satisfy only the displacement end conditions.

For the case of the rod with left-hand end fixed and right-hand end free, we will require the functions $\gamma(x)$ of Eq. (9.34) to satisfy

$$\gamma_j(0) = 0 \qquad j = 1, 2, \ldots, n \tag{9.43}$$

We have dropped the force condition on the right-hand end, the second of Eqs. (9.35). The error in the forces per unit length acting along the rod is given by Eq. (9.36) as before. It is important to recognize that the terms in Eq. (9.36) represent forces directed to the right, taken as the positive sense. We can expect there to be a discrepancy in the forces acting on the element at the right-hand end of the rod. The tensile force near the end will approach the value

$$\begin{aligned} F &= EAU'|_{x=l} \\ &\doteq EA\sum_{j=1}^{n}\gamma_j'|_{x=l}p_j \end{aligned} \tag{9.44}$$

As shown in Fig. 9-10, the finite error E in the forces acting on an infinitesimal end element is given by

$$0 - EA\sum_{j=1}^{n}\gamma_j'(l)p_j = E \tag{9.45}$$

The inertia force acting on the element will be infinitesimal and is not shown.

We will require the errors $\epsilon(x)$ and E to be orthogonal to each of the shape functions $\gamma(x)$, as given by

$$\int_0^l \epsilon(x)\gamma_i(x)\,dx + E\gamma_i(l) = 0 \qquad i = 1, 2, \ldots, n, \tag{9.46}$$

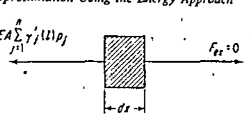

Fig. 9-10 *Forces acting on an element at the free right hand end of the rod.*

Substitution of Eqs (9.36) and (9.45) into Eqs. (9.46) leads to

$$\sum_{j=1}^{n} p_j\left[\int_0^l (EA\gamma_j')'\gamma_i\,dx - EA\gamma_j'(l)\gamma_i(l) + \omega^2\int_0^l \mu\gamma_i\gamma_j\,dx\right] = 0$$
$$i = 1, 2, \ldots, n \qquad (9.47)$$

Integration of the first integral of Eq. (9.47) by parts leads to Eq. (9.39) as before. From Eqs. (9.43), the integrated term of Eq. (9.39) evaluated at the left-hand end of the rod will be zero. Substitution of Eq. (9.39) into Eqs. (9.47) leads to

$$\sum_{j=1}^{n} p_j\left[-\int_0^l EA\gamma_i'\gamma_j'\,dx + \omega^2\int_0^l \mu\gamma_i\gamma_j\,dx\right] = 0 \qquad i = 1, 2, \ldots, n \qquad (9.48)$$

Two of the terms of Eqs. (9.39) and (9.47) have canceled. The result given by Eqs. (9.48) is identical with that given earlier by Eqs. (9.40) and (9.41)

Evidently, if the error in the forces at the ends of the rod is accounted for, the functions need satisfy only the displacement end conditions. This added flexibility in applying Galerkin's method becomes more important as more complicated systems are considered.

9.5 Approximation Using the Energy Approach—Method of Rayleigh-Ritz

In Secs 9.3 and 9.4 we have written the equations of motion in a finite number of coordinates using the method of collocation and Galerkin's method. In this section we will write the equations of motion in a finite set of generalized coordinates using work and energy methods.

To illustrate the procedure, let us consider the free longitudinal vibrations of an elastic rod. We can approximate the motion with n generalized coordinates p according to

$$u(x,t) = \sum_{j=1}^{n} \gamma_j(x)p_j(t) \qquad (9.49)$$

Consider the specific case of a rod fixed at the left-hand end and free at the right-hand end. Since the displacement conditions must be satisfied by the

assumed motion, we will require each of the shape functions to satisfy $\gamma_j(0) = 0$. If it is convenient, the shape functions are usually chosen to satisfy force conditions at the ends, in this case $EA\gamma_j'|_{x=l} = 0$.

Let us write the equations of motion from the kinetic and potential energy expressions. For the longitudinal motion of the rod, we can write the kinetic energy as

$$T = \frac{1}{2}\int_0^l \mu\dot{u}^2\,dx \tag{9.50}$$

Substitution of Eq. (9.49) into Eq. (9.50) leads to

$$\begin{aligned} T &= \frac{1}{2}\int_0^l \mu\left(\sum_{i=1}^{n} \gamma_i\dot{p}_i\right)\left(\sum_{j=1}^{n} \gamma_j\dot{p}_j\right) dx \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \dot{p}_i\dot{p}_j \int_0^l \mu\gamma_i\gamma_j\,dx \end{aligned}$$

Defining the generalized masses M by

$$M = \int_0^l \mu\gamma_i\gamma_j\,dx \tag{9.51}$$

we can write the kinetic energy of the rod as

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}\dot{p}_i\dot{p}_j \tag{9.52}$$

In matrix form, the kinetic energy is

$$T = \tfrac{1}{2}\lfloor\dot{p}\rfloor[M]\{\dot{p}\} \tag{9.53}$$

with

$$[M] = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & \cdot \\ \vdots & & & \vdots \\ M_{n1} & \cdot & \cdots & M_{nn} \end{bmatrix} \tag{9.54}$$

Then the generalized inertia forces are

$$\{P\}_{in} = -\frac{d}{dt}\left\{\frac{\partial T}{\partial \dot{p}}\right\} = -[M]\{\ddot{p}\} \tag{9.55}$$

For a free vibration in a single mode of vibration of frequency ω

$$\{\ddot{p}\} = -\omega^2\{p\}$$

The generalized inertia forces, Eq. (9.55), can be written as

$$\{P\}_{in} = \omega^2[M]\{p\} \tag{9.56}$$

The strain energy resulting from the longitudinal motion of the rod is

$$U = \frac{1}{2}\int_0^l EAu'^2\,dx \tag{9.57}$$

Substitution of the assumed motion, Eq. (9.49), into Eq. (9.57) leads to

$$\begin{aligned} U &= \frac{1}{2}\int_0^l EA\left(\sum_{i=1}^{n}\gamma_i' p_i\right)\left(\sum_{j=1}^{n}\gamma_j' p_j\right)dx \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j \int_0^l EA\gamma_i'\gamma_j'\,dx \end{aligned}$$

If the generalized stiffnesses K are defined by

$$K_{ij} = \int_0^l EA\gamma_i'\gamma_j'\,dx \tag{9.58}$$

the potential energy is

$$U = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} K_{ij}p_i p_j \tag{9.59}$$

In matrix form, we can write

$$U = \tfrac{1}{2}\lfloor p \rfloor[K]\{p\} \tag{9.60}$$

in which

$$[K] = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & \cdot \\ \vdots & & & \vdots \\ K_{n1} & & \cdots & K_{nn} \end{bmatrix} \tag{9.61}$$

The generalized elastic forces are given by

$$\{P\}_{el} = -\left\{\frac{\partial U}{\partial p}\right\} = -[K]\{p\} \tag{9.62}$$

From Eqs. (9.56) and (9.62), we can write the equations of motion for a free vibration as

$$\{\sum P\} = [\omega^2[M] - [K]]\{p\} = \{0\} \tag{9.63}$$

Note that this result compares with that given by Galerkin's method, Eqs (9.42). The method just outlined is referred to as the Rayleigh-Ritz method.

Normalizing to unity at the right-hand end of the rod, the approximate fundamental mode shape is

$$\phi_1(x) = 1.83\frac{x}{l} - 0.83\left(\frac{x}{l}\right)^2$$

For comparison, the exact fundamental mode shape given in Example 8.3 is

$$\phi_1(x) = \sin\frac{\pi x}{2l}$$

The exact fundamental natural frequency was given as

$$\omega_1 = \frac{\pi}{2}\sqrt{\frac{EA}{\mu l^2}} = 1.571\sqrt{\frac{EA}{\mu l^2}}$$

Our approximate value of $\omega_1 = 1.577\sqrt{\dfrac{EA}{\mu l^2}}$ is only 0.38 per cent too high.

EXAMPLE 9.8

Let us consider the continuous elastic rod of Fig. 8-16 which carries a mass m at the right-hand end. The kinetic energy of the rod in motion, given earlier by Eq. (9.50), must be modified to

$$T = \frac{1}{2}\int_0^l \mu \dot{u}^2\,dx + \tfrac{1}{2}m\dot{u}^2(l,t)$$

Substitution of approximate representation of the motion, Eq. (9.49), into the kinetic energy expression leads to

$$T \approx \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\dot{p}_i\dot{p}_j\left[\int_0^l \mu\gamma_i\gamma_j\,dx + m\gamma_i(l)\gamma_j(l)\right]$$

If we define the generalized masses by

$$M_{ij} = \int_0^l \mu\gamma_i\gamma_j\,dx + m\gamma_i(l)\gamma_j(l)$$

the form of the kinetic energy is given by Eq. (9.53) as before. The generalized stiffnesses, Eq. (9.58), are unchanged as are the equations of motion, Eq. (9.63).

Let us approximate the motion of the rod with the first two terms of a power series involving the shape functions

$$\gamma_1(x) = \frac{x}{l}$$

$$\gamma_2(x) = \left(\frac{x}{l}\right)^2$$

Normalizing to unity at the right-hand end of the rod, the approximate fundamental mode shape is

$$\phi_1(x) = 1.83\frac{x}{l} - 0.83\left(\frac{x}{l}\right)^2$$

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Substitution of approximate representation of the motion, Eq. (9.49), into the kinetic energy expression leads to

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j\left[\int_0^l \mu\gamma_i\gamma_j\,dx + m\gamma_i(l)\gamma_j(l)\right]$$

If we define the generalized masses by

$$M_{ij} = \int_0^l \mu\gamma_i\gamma_j\,dx + m\gamma_i(l)\gamma_j(l)$$

the form of the kinetic energy is given by Eq (9 53) as before The generalized stiffnesses, Eq. (9 58), are unchanged as are the equations of motion, Eq. (9.63)

Let us approximate the motion of the rod with the first two terms of a power series involving the shape functions

$$\gamma_1(x) = \frac{x}{l}$$

$$\gamma_2(x) = \left(\frac{x}{l}\right)^2$$

Normalizing to unity at the right-hand end of the rod, the approximate fundamental mode shape is

$$\phi_1(x) \doteq 1.83\frac{x}{l} - 0.83\left(\frac{x}{l}\right)^2$$

For comparison, the exact fundamental mode shape given in Example 8.3 is

$$\phi_1(x) = \sin\frac{\pi x}{2l}$$

The exact fundamental natural frequency was given as

$$\omega_1 = \frac{\pi}{2}\sqrt{\frac{EA}{\mu l^2}} = 1.571\sqrt{\frac{EA}{\mu l^2}}$$

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$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \dot{p}_i\dot{p}_j\left[\int_0^l \mu\gamma_i\gamma_j\, dx + m\gamma_i(l)\gamma_j(l)\right]$$

If we define the generalized masses by

$$M_{ij} = \int_0^l \mu\gamma_i\gamma_j\, dx + m\gamma_i(l)\gamma_j(l)$$

the form of the kinetic energy is given by Eq. (9.53) as before. The generalized stiffnesses, Eq. (9.58), are unchanged as are the equations of motion, Eq. (9.63).

Let us approximate the motion of the rod with the first two terms of a power series involving the shape functions

$$\gamma_1(x) = \frac{x}{l}$$

$$\gamma_2(x) = \left(\frac{x}{l}\right)^2$$

Then the generalized masses and stiffnesses are

$$M_{11} = \int_0^l \mu\left(\frac{x}{l}\right)^2 dx + m = \tfrac{1}{3}\mu l + m$$
$$M_{22} = \int_0^l \mu\left(\frac{x}{l}\right)^4 dx + m = \tfrac{1}{5}\mu l + m$$
$$M_{12} = M_{21} = \int_0^l \mu\left(\frac{x}{l}\right)^3 dx + m = \tfrac{1}{4}\mu l + m$$
$$K_{11} = \int_0^l EA\frac{1}{l^2}\,dx = \frac{EA}{l}$$
$$K_{22} = \int_0^l EA\frac{4}{l^2}\left(\frac{x}{l}\right)^2 dx = \frac{4}{3}\frac{EA}{l}$$
$$K_{12} = K_{21} = \int_0^l EA\frac{2}{l^2}\frac{x}{l}\,dx = \frac{EA}{l}$$

Consider the special case in which the masses of the rod and of the end mass are equal, as given by $\dfrac{\mu l}{m} = 1$. The equations of motion, Eqs. (9.63), are

$$\left[\mu l\omega^2\begin{bmatrix} \frac{4}{3} & \frac{5}{4} \\ \frac{5}{4} & \frac{6}{5} \end{bmatrix} - \frac{EA}{l}\begin{bmatrix} 1 & 1 \\ 1 & \frac{4}{3} \end{bmatrix}\right]\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Solving for the two eigenvalues, we can write

$$\omega_1^2 = \frac{20}{27}\frac{EA}{\mu l^2} = 0.7407\frac{EA}{\mu l^2}$$
$$\omega_2^2 = 12\frac{EA}{\mu l^2}$$

The relative amplitudes of the coordinates in the two modes are

$$\frac{p_{11}}{p_{21}} = -6$$
$$\frac{p_{12}}{p_{22}} = -\frac{14}{15} = -0.933$$

Then the fundamental mode normalized to unity at the right-hand end is given by

$$\phi_1(x) = 1.2\frac{x}{l} - 0.2\left(\frac{x}{l}\right)^2$$

As a comparison, the exact fundamental mode shape is given by Example 8.8 as

$$\phi_1(x) = \sin 0.860\frac{x}{l}$$

Note that this result is not normalized to unity at the right-hand end. The exact fundamental natural frequency was given as $\omega_1 = 0.860\sqrt{\frac{EA}{\mu l^2}}$. To three significant figures our approximate result is $\omega_1 = 0.861\sqrt{\frac{EA}{\mu l^2}}$.

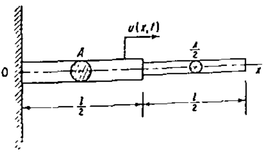
Problems

9-1 Write the influence function for the torsional displacement of a uniform elastic rod fixed at both ends. Obtain the integral equation for torsional vibration of the rod. Using the separation of variables technique, write the integral equation which the eigenfunctions and eigenvalues must satisfy.

9-2 For the flexible string of Prob. 8-1, write the influence function for small transverse displacements in a plane. Give the integral equation for a free transverse vibration of small amplitude of the string.

9-3 Write the influence function for the flexural displacement of a simply supported uniform elastic beam. Obtain the integral equation for the free flexural vibrations of the beam. Using the separation of variables technique, write the integral equation in the eigenfunctions and eigenvalues

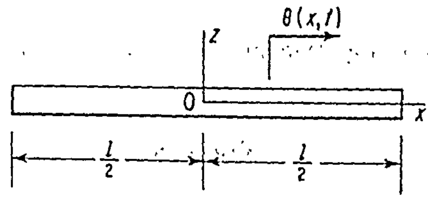
9-4 An elastic rod is made up of two uniform sections with cross sectional areas A and $A/2$ as shown. Determine the influence function for longitudinal displacement of the rod. Give the integral equation for the longitudinal vibrations of the rod.



Prob. 9-4

9-5 Consider the torsional motion of the uniform elastic rod having free ends as shown. Obtain the influence function for the elastic torsional displacement of the rod relative to the frame of reference oxz which is attached to the rod at O. Write the integral equation for the free torsional vibrations of the rod relative to oxz and the equation for the accompanying torsional

motion of oxz. Note that the angular momentum of the rod around its axis must be constant in a free vibration.



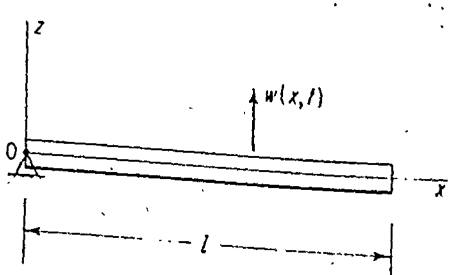
Prob. 9-5

9-6 In Prob. 9-5 the rotation $\theta(x,t)$ of the rod can be written as $\theta(x,t) = \theta_R(t) + \theta_E(x,t)$ in which $\theta_R(t)$ represents the rotation of oxz and $\theta_E(x,t)$ represents the rotation of the rod relative to oxz. Instead of fixing oxz to the rod at O, let us require that

$$\int_{-l/2}^{l/2} I\theta_E \, dx = 0$$

Thus oxz is chosen such that the angular momentum of the rod relative to oxz is always zero. For a free torsional vibration, show that $\dot{\theta}_R$ is constant. Write the influence function for the elastic torsional displacement of the rod relative to oxz. Obtain the integral equation for the free torsional vibrations of the rod relative to oxz.

9-7 A uniform elastic beam is pinned at one end and free at the other as shown. Suppose the frame of reference oxz is fixed at O with ox tangent to the beam at O. Write the influence function for the flexural displacement of the beam relative to oxz resulting from transverse forces applied to the beam. Give the integral equation for the free flexural vibrations of the beam relative to oxz. Write the equation for the accompanying angular motion of oxz. The angular momentum of the rod around O must be constant in a free vibration.



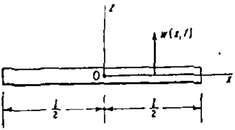
Prob. 9-7

9-8 The motion of the beam of Prob. 9-7 is completely described by the rotation $\theta_R(t)$ of oxz and the flexural displacement $w_E(x,t)$ of the beam relative to oxz. Let us require the orientation of oxz to be consistent with

$$\int_0^l \mu x w_E \, dx = 0$$

According to this requirement, oxz is chosen such that the angular momentum of the beam around O and relative to oxz is always zero. Note that $\dot{\theta}_R$ is constant for a free flexural vibration. Determine the influence function for the flexural displacement of the beam relative to oxz resulting from transverse forces applied to the beam. Give the integral equation for the free flexural vibrations of the beam relative to oxz.

9-9 A uniform elastic beam is free at both ends as shown. The flexural motion of the beam is completely described by the displacement and rotation



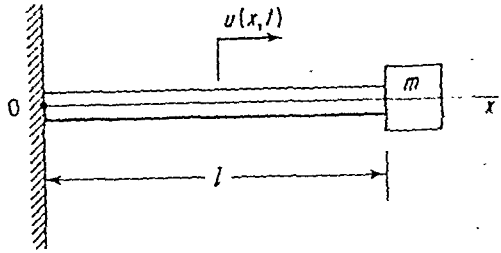
Prob. 9-9

$w_R(t)$ and $\theta_R(t)$ of the frame of reference oxz and the flexural displacement $w_E(x,t)$ of the beam relative to oxz. It is convenient to require the position and orientation of oxz to comply with

$$\int_{-l/2}^{l/2} \mu w_E \, dx = 0$$
$$\int_{-l/2}^{l/2} \mu x w_E \, dx = 0$$

Thus oxz is chosen such that the linear momentum and the angular momentum around O of the beam relative to oxz are always zero. Write the influence function for the flexural displacement of the beam relative to oxz resulting from transverse forces applied to the beam. Obtain the integral equation for the free flexural vibration of the beam relative to oxz.

9-10 Consider the uniform rod with mass m attached as shown. Write the influence function for longitudinal displacements of the system. Give the integral equation for the longitudinal motion of the system. Following the technique of separation of variables, write the integral equation in the eigenfunctions and eigenvalues.



Prob. 9-10

9-11 A mass m is fixed to the midpoint of the flexible string of Prob. 8-1. Considering the string to be uniform, write the integral equation for transverse vibrations of small amplitude.

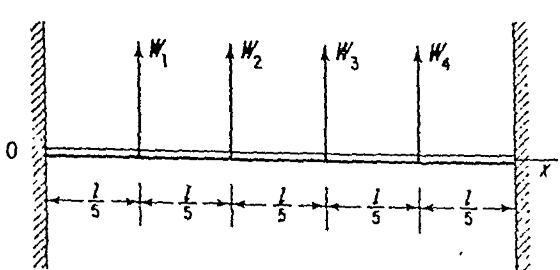
9-12 Determine the fundamental eigenvalue and eigenfunction of the system of Prob. 9-4 using the method of direct iteration. Begin with the trial eigenfunction $W(x) = x/l$.

9-13 Obtain the fundamental eigenvalue and eigenfunction of the simply supported uniform beam considered in Prob. 9-3. Use the method of direct iteration along with the integral equation. The suggested trial eigenfunction $W(x) = 4\frac{x}{l}\left(1 - \frac{x}{l}\right)$ has been normalized such that $W\left(\frac{l}{2}\right) = 1$.

9-14 Use the integral equation of Prob. 9-8 along with the method of direct iteration to obtain the eigenvalue and eigenfunction for the first elastic mode of a uniform pinned-free beam.

9-15 Obtain the fundamental eigenvalue and eigenfunction of the system of Prob. 9-10, using the method of direct iteration. Begin with the trial eigenfunction $U(x) = x/l$. The mass μl of the rod equals the mass m.

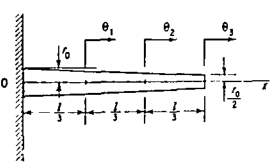
9-16 The free transverse vibrations of a uniform string are to be described by the displacements W_1, W_2, W_3, W_4 at the four equally spaced stations shown. Using the integral equation of Prob. 9-2 and the methods of collocation and numerical integration, write the four equations of motion. In the



Prob. 9-16

numerical integration use the trapezoidal rule Iterate to obtain the fundamental natural frequency and normal mode shape Compare the results with those of Prob. 8-8.

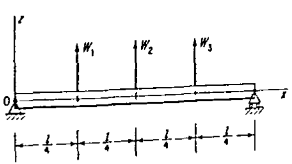
9-17 A solid tapered rod has a radius which varies linearly from r_0 at the root to $r_0/2$ at the tip as shown. The rotation of the rod can be approximated in terms of rotations Θ_1, Θ_2, Θ_3. Apply the methods of collocation



Prob. 9-17

and numerical integration to the integral equation to obtain the three equations of motion. Use the trapezoidal rule in the integration. Iterate to obtain the fundamental natural frequency and normal mode shape.

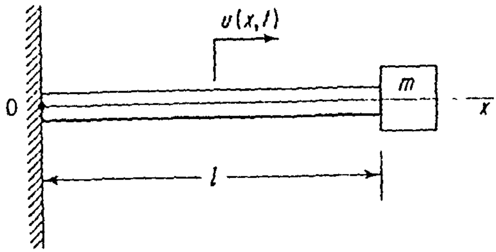
9-18 The free flexural vibrations of the uniform simply supported beam shown are to be represented by the displacements W_1, W_2, W_3 Write the three equations of motion by applying the methods of collocation and



Prob. 9-18

numerical integration to the integral equation If the trapezoidal rule is used in the integration, the equations of motion will be identical with those of Prob. 5-9 with $m = \mu l/4$.

9-19 Let us represent the free longitudinal vibration of the system of Prob. 9-10 with the coordinates U_1, U_2, U_3 as shown. Apply the methods of collocation and numerical integration to the integral equation of motion to



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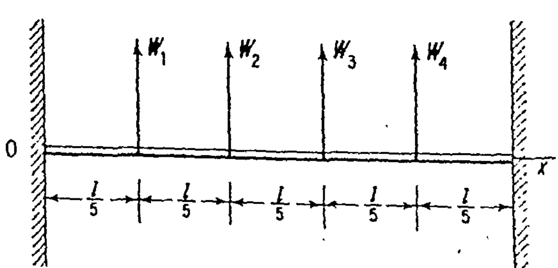
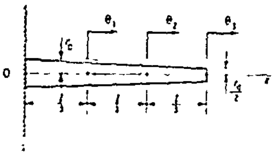


Prob. 9-16

numerical integration use the trapezoidal rule. Iterate to obtain the fundamental natural frequency and normal mode shape. Compare the results with those of Prob 8-8.

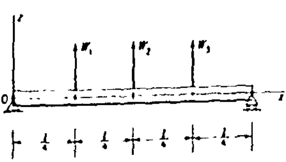
9-17 A solid tapered rod has a radius which varies linearly from r_0 at the root to $r_0/2$ at the tip as shown. The rotation of the rod can be approximated in terms of rotations θ_1, θ_2, θ_3. Apply the methods of collocation



Prob. 9-17

and numerical integration to the integral equation to obtain the three equations of motion. Use the trapezoidal rule in the integration. Iterate to obtain the fundamental natural frequency and normal mode shape.

9-18 The free flexural vibrations of the uniform simply supported beam shown are to be represented by the displacements W_1, W_2, W_3. Write the three equations of motion by applying the methods of collocation and



Prob 9-18

numerical integration to the integral equation. If the trapezoidal rule is used in the integration, the equations of motion will be identical with those of Prob 5-9 with $m = \mu l/4$

9-19 Let us represent the free longitudinal vibration of the system of Prob 9-10 with the coordinates U_1, U_2, U_3 as shown. Apply the methods of collocation and numerical integration to the integral equation of motion to

9-26 Let us represent the rotation of the tapered rod of Prob. 9-17 with three coordinates having shape functions

$$\gamma_1(x) = \frac{x}{l}$$
$$\gamma_2(x) = \left(\frac{x}{l}\right)^2$$
$$\gamma_3(x) = \left(\frac{x}{l}\right)^3$$

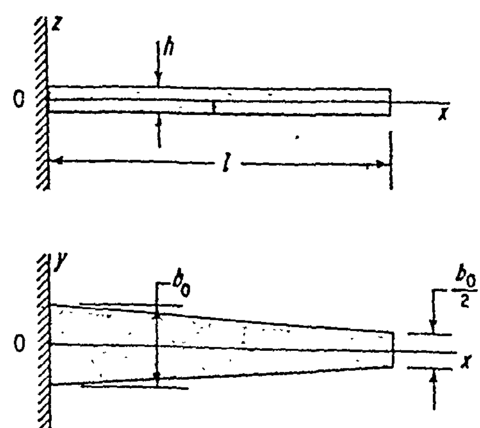
Write the equations of motion using the energy approach. Solve for the fundamental natural frequency and normal mode shape.

9-27 We can approximate the longitudinal motion of the system of Prob. 9-10 with two coordinates having shape functions

$$\gamma_1(x) = \sin\frac{\pi x}{2l}$$
$$\gamma_2(x) = \sin\frac{3\pi x}{2l}$$

Following the Rayleigh-Ritz procedure, write the equations of motion for a free vibration of the system. Assume that the mass μl of the rod is equal to the mass m. Obtain an estimate for the fundamental natural frequency. Compare the result with that of Example 9.8.

9-28 A beam has a constant depth h but a width which varies linearly from b_0 at the root to $b_0/2$ at the tip as shown. We can approximate the free



Prob. 9-28

flexural vibration $w(x,t)$ of the beam with two coordinates having shape functions

$$\gamma_1(x) = \left(\frac{x}{l}\right)^2$$

$$\gamma_2(x) = \left(\frac{x}{l}\right)^3$$

Following the Rayleigh-Ritz procedure, write the equations of motion. Obtain an estimate for the fundamental natural frequency and normal mode shape of the beam.

9-26 Let us represent the rotation of the tapered rod of Prob. 9-17 with three coordinates having shape functions

$$\gamma_1(x) = \frac{x}{l}$$
$$\gamma_2(x) = \left(\frac{x}{l}\right)^2$$
$$\gamma_3(x) = \left(\frac{x}{l}\right)^3$$

Write the equations of motion using the energy approach. Solve for the fundamental natural frequency and normal mode shape.

9-27 We can approximate the longitudinal motion of the system of Prob. 9-10 with two coordinates having shape functions

$$\gamma_1(x) = \sin\frac{\pi x}{2l}$$
$$\gamma_2(x) = \sin\frac{3\pi x}{2l}$$

Following the Rayleigh-Ritz procedure, write the equations of motion for a free vibration of the system. Assume that the mass μl of the rod is equal to the mass m. Obtain an estimate for the fundamental natural frequency. Compare the result with that of Example 9.8.

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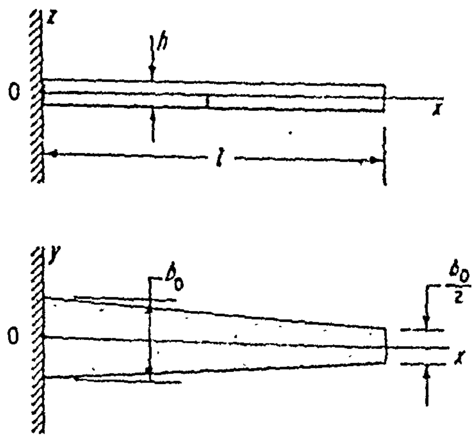


Prob. 9-28

flexural vibration $w(x,t)$ of the beam with two coordinates having shape functions

$$\gamma_1(x) = \left(\frac{x}{l}\right)^2$$

$$\gamma_2(x) = \left(\frac{x}{l}\right)^3$$

Following the Rayleigh-Ritz procedure, write the equations of motion. Obtain an estimate for the fundamental natural frequency and normal mode shape of the beam.

CHAPTER 10

Forced Vibrations of Continuous Systems

10.1 Equations of Motion with Force Excitation

Usually the disturbance which results in forced vibrations of a system is described most closely in terms of prescribed external forces. For the present, let us assume that the external exciting forces are given. In Sec. 10.5 we will consider situations in which the displacements at the points of excitation are given. It is generally most convenient to describe the motion of a system in terms of the normal modes of vibration. Thus, in the following it will be our regular practice to write the equations of motion in the normal coordinates. Other choices of coordinates can of course be made and the equations of motion written in a similar manner.

For the longitudinal vibrations of the elastic rod of Fig. 8-1, the exciting force is shown as a distributed longitudinal force $f_{ex}(x,t)$ acting along the rod. If the external forces are prescribed at the ends, they can either be zero, corresponding to a free end, or nonzero, indicating force excitation. If, instead, the displacements are prescribed at the ends, they can only be zero, corresponding to a fixed end, since we have ruled out displacement excitation. Thus, at the ends we can either specify a fixed or free end or give the external exciting force, as follows

$$\begin{aligned} u(0,t) &= 0 \qquad \text{or} \qquad F_{ex}(0,t) \\ u(l,t) &= 0 \qquad \text{or} \qquad F_{ex}(l,t) \end{aligned} \tag{10.1}$$

We can write the generalized forces associated with the normal coordinates by applying the principle of virtual displacements. As an illustration, let us consider the case in which the left-hand end is fixed and the right-hand end experiences a prescribed external force. In a virtual displacement of the ith normal coordinate, the resulting displacement of the rod is given by $\phi_i(x)\,\delta q_i$. The virtual work done by the external forces is

$$\begin{aligned} \delta W &= \int_0^l f_{ex}\phi_i\,\delta q_i\,dx + F_{ex}(l,t)\phi_i(l)\,\delta q_i \\ &= \delta q_i\left[\int_0^l f_{ex}\phi_i\,dx + F_{ex}(l,t)\phi_i(l)\right] \end{aligned}$$

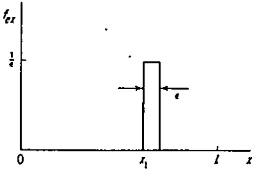


Fig. 10-1 *Unit concentrated force at $x = x_1$, $\epsilon \to 0$.*

Then the generalized external force associated with the ith normal coordinate q_i is

$$Q_{i,ex} = \int_0^l f_{ex}\phi_i \, dx + F_{ex}(l,t)\phi_i(l) \tag{10.2}$$

In addition to the distributed force f_{ex}, there may be concentrated forces acting at points other than the end points. If a force $F_1(t)$ acts at the point $x = x_1$, we must add a term $F_1\phi_i(x_1)$ to the expression for the generalized external force, Eq (10.2) Instead of considering concentrated forces separately, they may be treated as distributed forces by using the concept of the delta function. A unit concentrated force acting at the point $x = x_1$ can be visualized as a distributed force $1/\epsilon$ acting over the internal ϵ as shown in Fig. 10-1. The function shown with $\epsilon \to 0$ is generally called a delta function and identified by $\delta(x - x_1)$. We can write the concentrated force F_1 acting at $x = x_1$ as a distributed force by means of the expression $F_1\,\delta(x - x_1)$. Substitution of this expression for f_{ex} in the integral on the right-hand side of Eq. (10.2) leads to

$$\int_0^l F_1\,\delta(x - x_1)\phi_i(x)\,dx = F_1\phi_i(x_1)$$

which is the expected result. Adding the generalized external force, Eq. (10.2), to the equations of motion in the normal coordinates, given by Eq (8.70), leads to

$$\sum Q_i = -M_i\ddot{q} - K_iq_i + Q_{i,ex} = 0 \qquad i = 1, 2, . \tag{10.3}$$

The generalized masses M_i and stiffnesses K_i are given by Eqs. (8 63) and (8.67).

The discussion above is appropriate for the torsional vibrations of an elastic rod Let us suppose that a distributed torque $\tau_{ex}(x,t)$ acts along the

rod as shown in Fig. 8-2. At the ends of the rod, we can either specify a fixed or a free end or give the external exciting torque, as follows

$$\begin{aligned} \theta(0,t) &= 0 \quad \text{or} \quad T_{ex}(0,t) \\ \theta(l,t) &= 0 \quad \text{or} \quad T_{ex}(l,t) \end{aligned} \tag{10.4}$$

Applying the principle of virtual displacements, we can write the generalized external force associated with the ith normal coordinate q_i. From Eq. (10.2), the generalized external force for the case in which the left-hand end of the rod is fixed and the right-hand end experiences a prescribed torque is given by

$$Q_{i,ex} = \int_0^l \tau_{ex}\phi_i \, dx + T_{ex}(l,t)\phi_i(l) \tag{10.5}$$

A concentrated torque T_1 acting at $x = x_1$ can be represented as a distributed torque in the form $\tau_{ex} = T_1\,\delta(x - x_1)$. Thus the distributed torque τ_{ex} may include several concentrated torques. The equations of forced vibration in the normal coordinates have the form of Eqs. (10.3) in which the generalized masses and stiffnesses are given by Eqs. (8.75).

Consider the forced flexural vibrations of the elastic beam of Fig. 8-3. The exciting force is shown as a distributed transverse force $f_{ex}(x,t)$. For the most general force excitation, we should assume that a distributed bending moment $m_{ex}(x,t)$ also acts on the beam. If either the transverse or angular displacement of the beam is prescribed at an end of the beam, its magnitude must be zero since displacement excitation is not being considered. If the transverse displacement is not prescribed at an end, the shear force must be given. Similarly, the bending moment at an end must be given if the slope is not prescribed. The end conditions required are summarized by

$$\begin{aligned} w(0,t) &= 0 \quad \text{or} \quad S_{ex}(0,t) \\ w'(0,t) &= 0 \quad \text{or} \quad M_{ex}(0,t) \\ w(l,t) &= 0 \quad \text{or} \quad S_{ex}(l,t) \\ w'(l,t) &= 0 \quad \text{or} \quad M_{ex}(l,t) \end{aligned} \tag{10.6}$$

Let us consider the special case of a beam in which the left-hand end is fixed and the right-hand end experiences a prescribed shear force and bending moment. In a virtual displacement of the ith normal coordinate, the resulting transverse and angular displacements of the beam elements are given by $\phi_i(x)\,\delta q_i$ and $\phi_i'(x)\,\delta q_i$. The virtual work done by the external forces and moments is given by

$$\begin{aligned} \delta W &= \int_0^l (f_{ex}\phi_i + m_{ex}\phi_i')\,\delta q_i\,dx - S_{ex}(l,t)\phi_i(l)\,\delta q_i + M_{ex}(l,t)\phi_i'(l)\,\delta q_i \\ &= \delta q_i\left[\int_0^l (f_{ex}\phi_i + m_{ex}\phi_i')\,dx - S_{ex}(l,t)\phi_i(l) + M_{ex}(l,t)\phi_i'(l)\right] \end{aligned}$$

The minus sign on the right-hand side of this equation results from the convention that positive shear force on the right-hand end of the beam is directed downward. We can write the generalized external force associated with the ith normal coordinate as

$$Q_{i,ex} = \int_0^l (f_{ex}\phi_i + m_{ex}\phi_i')\,dx - S_{ex}(l,t)\phi_i(l) + M_{ex}(l,t)\phi_i'(l) \quad (10.7)$$

Concentrated transverse forces or bending moments can be represented as distributed transverse forces or bending moments respectively by using the concept of the delta function. For illustration, we can write the concentrated bending moment M_1 acting at $x = x_1$ as $M_1\,\delta(x - x_1)$. The equations of forced vibration in the normal coordinates will have the form of Eqs. (10.3). The generalized masses and stiffnesses involved in the equations are given by Eqs. (8.82)

In most real problems, it is more than likely that the normal modes of the system have been determined using an approximate method such as one of those outlined in Chap. 9 The normal mode shapes obtained with the Galerkin or Rayleigh-Ritz methods are each given by a finite series of selected functions. Although the mode shapes are approximate, they are mutually orthogonal Thus the equations of motion in the normal coordinates are uncoupled and are given, as before, by Eqs (10.3). Further, the expressions for the generalized masses, stiffnesses and exciting forces given for the exact normal modes are still appropriate. The normal mode shapes obtained with the method of collocation, using numerical methods, are each given by a column of numbers representing displacements at selected points of the system. With the normal mode shapes given in this form it is necessary to evaluate the integral expressions for the generalized masses, stiffnesses and exciting forces using numerical integration techniques

EXAMPLE *10.1*

A uniform rod, constrained as shown in Fig. 10-2, is excited by longitudinal forces, a concentrated force $F(t)$ acting at the midpoint and a uniform distributed force $f(t)$ acting along the rod. From Example 8.3, the exact normal mode shapes for the rod are

$$\phi_i(x) = \sin\frac{(2i-1)\pi x}{2l} \qquad i = 1, 2,$$

Using Eqs. (8.63) and (8.67), we can write the generalized masses and stiffnesses as

$$K_i = \int_0^l EA\,\frac{(2i-1)^2\pi^2}{4l^2}\cos^2\frac{(2i-1)\pi}{2l}x\,dx = \frac{(2i-1)^2\pi^2}{8}\,\frac{EA}{l}$$

$$M_i = \int_0^l \mu\sin^2\frac{(2i-1)\pi x}{2l}\,dx = \frac{\mu l}{2}$$

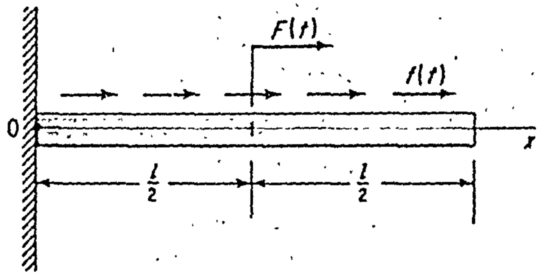


Fig. 10-2

Let us represent the concentrated force $F(t)$ as a distributed force by the expression $F\,\delta\left(x - \frac{l}{2}\right)$. From Eq. (10.2), we can write the generalized external forces associated with the normal coordinates as

$$\begin{aligned}
Q_{i,ex} &= \int_0^l \left[f + F\,\delta\left(x - \frac{l}{2}\right)\right] \sin\frac{(2i-1)\pi x}{2l}\,dx \\
&= -\frac{2fl}{(2i-1)\pi}\cos\frac{(2i-1)\pi x}{2l}\Big|_0^l + F\sin\frac{(2i-1)\pi}{4} \\
&= \frac{2}{(2i-1)\pi} fl + F\sin\frac{(2i-1)\pi}{4}
\end{aligned}$$

The equations of motion will have the form of Eqs. (10.3), as given by

$$\begin{aligned}
&\frac{\mu l}{2}\ddot{q}_1 + \frac{\pi^2}{8}\cdot\frac{EA}{l}q_1 = \frac{2}{\pi}fl + \frac{\sqrt{2}}{2}F \\
&\frac{\mu l}{2}\ddot{q}_2 + \frac{9\pi^2}{8}\cdot\frac{EA}{l}q_2 = \frac{2}{3\pi}fl + \frac{\sqrt{2}}{2}F \\
&\vdots
\end{aligned}$$

EXAMPLE *10.2*

In Example 9.4 the methods of collocation and numerical integration were used to write the equations of motion for the uniform rod of Fig. 9-6. The longitudinal motion of the rod is described in terms of the motion of the four equally spaced points as shown. Solution of the equations of motion yields the fundamental normal mode, described by

$$\omega_1^2 = 2.43\,\frac{EA}{\mu l^2}$$

$$\{\phi\}_1 = \begin{Bmatrix} 0.393 \\ 0.707 \\ 0.923 \\ 1 \end{Bmatrix}$$

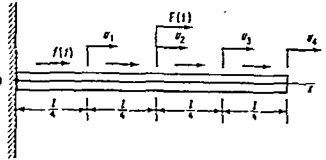


Fig. 10-3

Assuming that the rod is excited by the forces shown in Fig. 10-2, let us write the equation of motion in the approximate fundamental normal mode given. The longitudinal motion of the rod can be described in terms of the four coordinates u_1, u_2, u_3, u_4 as shown in Fig. 10-3. Referring to the normal mode shape, we can write

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0.393 \\ 0.707 \\ 0.923 \\ 1 \end{Bmatrix} q_1$$

in which q_1 is the fundamental normal coordinate.

From Eq. (8 63) and (10 2), the generalized mass and external forces are given by

$$M_1 = \int_0^l \mu\phi_1^2\,dx$$

$$Q_{1,ex} = \int_0^l \left[f + F\delta\left(x - \frac{l}{2}\right)\right]\phi_1\,dx$$

$$= \int_0^l f\phi_1\,dx + F\phi_1\left(\frac{l}{2}\right)$$

The magnitude of the mode shape function $\phi_1(x)$ is known at the origin and at the four equally spaced points. We can write

$$\phi_1(0) = \phi_{01} = 0$$

$$\phi_1\left(\frac{l}{4}\right) = \phi_{11} = 0.393$$

$$\phi_1\left(\frac{l}{2}\right) = \phi_{21} = 0.707$$

$$\phi_1\left(\frac{3l}{4}\right) = \phi_{31} = 0.923$$

$$\phi_1(l) = \phi_{41} = 1$$

Following a numerical integration procedure,

$$M_1 = \mu \sum_{i=1}^{4} \phi_{i1}^2 w_i$$

$$Q_{1,ex} = f \sum_{i=1}^{4} \phi_{i1} w_i + F\phi_{21}$$

in which the quantities w are the weighting numbers. Using the trapezoidal rule, the weighting numbers are given by

$$w_1 = w_2 = w_3 = \frac{l}{4}$$

$$w_4 = \frac{l}{8}$$

and the generalized mass and external force by

$$M_1 = 0.502\mu l$$
$$Q_{1,ex} = 0.631 fl + 0.707F$$

Then the generalized stiffness is

$$K_1 = M_1\omega_1^2 = 1.22\frac{EA}{l}$$

We can write the equation of motion in the fundamental normal coordinate as

$$0.502\mu l\ddot{q}_1 + 1.22\frac{EA}{l}q_1 = 0.631fl + 0.707F$$

Note that this result compares closely with the equation written for the exact fundamental normal coordinate given in Example 10.1.

EXAMPLE *10.3*

Consider the uniform elastic rod of Fig. 8-16. Suppose, as shown in Fig. 10-4, that the rod is excited by a concentrated longitudinal force $F(t)$ acting on the mass m. In Example 9.8 the Rayleigh-Ritz method is used to obtain

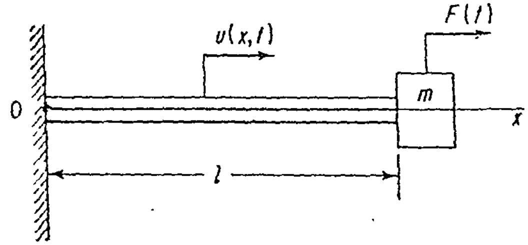


Fig. 10-4

the approximate mode shapes for two normal modes. For the special case in which the mass of the rod equals the end mass m, given by $\mu l/m = 1$, the approximate fundamental normal mode shape is

$$\phi_1(x) = 1.2\frac{x}{l} - 0.2\left(\frac{x}{l}\right)^2$$

Let us then write the equation of motion in this approximate mode. From Eqs. (8.67) and (8.89), we can write the generalized mass and stiffness as

$$\begin{aligned} M_1 &= \mu \int_0^l \left[1.2\frac{x}{l} - 0.2\left(\frac{x}{l}\right)^2\right]^2 dx + m(1)^2 \\ &= 0.368\mu l + m = 1.368\mu l \\ K_1 &= EA \int_0^l \frac{1}{l^2}\left[1.2 - 0.4\frac{x}{l}\right]^2 dx \\ &= 1.013\frac{EA}{l} \end{aligned}$$

From Eq. (10.2), the generalized external force is

$$Q_{1,ex} = F\phi_1(l) = F$$

Then the equation of motion in the fundamental normal coordinate may be written as

$$1.368\mu l\ddot{q}_1 + 1.013\frac{EA}{l}q_1 = F$$

Given the time-history of $F(t)$, we can determine the response using methods appropriate for single-degree-of-freedom systems.

10 2 Solution for the Motion and the Resulting Elastic Forces

We can solve the uncoupled equations of motion in the normal coordinates using the methods outlined in Chap 3 for single-degree-of-freedom systems. Substitution of the solution for the normal coordinates into the equation of transformation, Eq. (8 42), yields the motion solution. This procedure is referred to as the mode-displacement method.

If we wish to determine the internal elastic force at a selected point of a system, we will first need to establish the elastic force resulting from a unit displacement of each of the normal coordinates. Let us represent the internal elastic force per unit displacement of the ith normal coordinate by l_i. For the ith normal mode of longitudinal vibration in a rod, the longitudinal elastic force for a unit displacement is

$$l_i = EA\phi_i' \tag{10 8}$$

Similarly, the elastic torque in a rod for a unit displacement of the ith normal mode of torsional vibration is

$$l_i = GJ\phi_i' \tag{10.9}$$

For the ith normal mode of flexural vibration in a beam, the elastic bending moment and shear force resulting from a unit displacement are

$$\begin{aligned} l_i &= EI\phi_i'' \\ l_i &= (EI\phi_i'')' \end{aligned} \tag{10.10}$$

Superimposing the contributions of all of the normal modes considered, we can write

$$L = \sum_{i=1}^{n} l_i q_i \tag{10.11}$$

in which L represents the total internal elastic force or moment at the selected point. This method for determining the internal elastic forces is referred to as the mode-displacement method. If the number n of the normal modes included in Eq. (10.11) is finite, the result will of course be approximate.

If the normal mode shapes are obtained using the method of collocation along with numerical methods, the modal amplitudes will be given only at selected points of the system. In this case, it will be necessary to use a finite difference method to represent the derivatives of Eqs. (10.8) through (10.10).

EXAMPLE *10.4*

Let us suppose that the exciting forces acting on the system of Fig. 10-2 are harmonic with frequency ω, as given by

$$\begin{aligned} f &= f_0 \sin \omega t \\ F &= F_0 \sin \omega t \end{aligned}$$

Then the steady-state solution for the equations of motion given in Example 10.1 is

$$q_i = \frac{\dfrac{2}{(2i-1)\pi} f_0 l + F_0 \sin \dfrac{(2i-1)\pi}{4}}{\dfrac{(2i-1)^2\pi^2}{8} \cdot \dfrac{EA}{l} - \dfrac{\mu l}{2}\omega^2} \sin \omega t \qquad i = 1, 2, \ldots$$

From the results of Example 8.3, the natural frequencies are

$$\omega_i = \frac{(2i-1)\pi}{2} \sqrt{\frac{EA}{\mu l^2}}$$

We can write the solution for the steady-state motion of the normal coordinates as

$$q_i = \frac{8}{(2i-1)^2\pi^2}\frac{l}{EA}\frac{\frac{2}{(2i-1)\pi}f_0 l + F_0 \sin\frac{(2i-1)\pi}{4}}{1-\frac{\omega^2}{\omega_i^2}}\sin\omega t \qquad i = 1, 2, \ldots$$

Superimposing the contributions of the first n of the normal modes, we can write the solution for the steady-state longitudinal motion of the rod as

$$u(x,t) = \frac{l}{EA}\sum_{i=1}^{n}\frac{8}{(2i-1)^2\pi^2}\frac{\frac{2}{(2i-1)\pi}f_0 l + F_0 \sin\frac{(2i-1)\pi}{4}}{1-\frac{\omega^2}{\omega_i^2}} \times \sin\frac{(2i-1)\pi x}{2l}\sin\omega t$$

We have used the natural mode shapes, given in Example 8.3 as

$$\phi_i(x) = \sin\frac{(2i-1)\pi x}{2l}$$

Any one of the normal modes can be put in a resonant condition by letting the driving frequency approach the natural frequency. In the higher modes, for which the natural frequency is higher than the driving frequency, the quantity $1 - \frac{\omega^2}{\omega_i^2}$ approaches unity. Then the displacement response in the higher modes to the distributed and concentrated exciting forces converge as i^{-3} and i^{-2} respectively. As a result, not many of the higher modes need to be included in the displacement response. It is characteristic that the convergence is more rapid with distributed than with concentrated force excitation.

From Eq. (10.8), the elastic force in the rod per unit displacement of the ith normal coordinate is given by

$$l_i = \frac{(2i-1)\pi}{2}\cdot\frac{EA}{l}\cos\frac{(2i-1)\pi x}{2l}$$

Using Eq. (10.11), we can write the solution for the steady-state force in the rod as

$$L = \sum_{i=1}^{n}\frac{4}{(2i-1)\pi}\frac{\frac{2}{(2i-1)\pi}f_0 l + F_0 \sin\frac{(2i-1)\pi}{4}}{1-\frac{\omega^2}{\omega_i^2}}\cos\frac{(2i-1)\pi x}{2l}\sin\omega t$$

Evidently the response in the elastic force to the distributed and concentrated exciting forces converge as i^{-2} and i^{-1} respectively. The force response is less convergent than the displacement response and more terms are needed

Substitution of the results of Eq. (10.19) into the second of Eqs. (10.14) leads us to the approximate solution for u_{in}, given by

$$u_{in} = -\sum_{i=1}^{n} \phi_i \frac{\ddot{q}_i}{\omega_i^2} \tag{10.20}$$

We can write the approximate solution for longitudinal displacement of the rod as

$$u = u_{ex} - \sum_{i=1}^{n} \phi_i \frac{\ddot{q}_i}{\omega_i^2} \tag{10.21}$$

It is assumed that u_{ex} has been determined independently of the vibration analysis and is an exact result. The method outlined is referred to as the mode-acceleration method.

Let us compare the results of the mode-acceleration method with those of the mode-displacement method. According to the mode-displacement method, the longitudinal displacement of the rod is approximated by

$$u = \sum_{i=1}^{n} \phi_i q_i \tag{10.22}$$

or, using Eq. (10.13), by

$$u = \sum_{i=1}^{n} \phi_i q_{i,ex} + \sum_{i=1}^{n} \phi_i q_{i,in} \tag{10.23}$$

From Eq. (10.19) the second terms on the right-hand sides of Eqs. (10.21) and (10.23) are identical. The first terms on the right-hand sides of Eqs. (10.21) and (10.23), representing the displacement response to the external forces, are different. They differ in that the term in Eq. (10.21) is exact while the term in Eq. (10.23) is approximate, assuming the number n of the normal coordinates is finite. Thus the mode-acceleration method represents an improvement over the mode-displacement method.

Very often, the displacement solution is quite convergent with the mode number and the mode-displacement method is adequate. However, the solution for the internal elastic forces will in general be less convergent and the choice of methods is likely to be important. Let us separate the internal elastic force L at a point of the system into two parts, the response L_{ex} to the external forces and the response L_{in} to the inertia forces, as in

$$L = L_{ex} + L_{in} \tag{10.24}$$

In many cases, we will be able to obtain the exact response L_{ex} to the external forces using the methods of statics along with the methods of mechanics of

materials. Letting l_i represent the internal elastic force per unit displacement of the normal coordinate q_i, we can write the solution for L_{in} as

$$L_{in} = \sum_{i=1}^{n} l_i q_{i,in} \tag{10.25}$$

or, using Eq. (10.19), as

$$L_{in} = -\sum_{i=1}^{n} l_i \frac{\ddot{q}_i}{\omega_i^2} \tag{10.26}$$

Then we can write the approximate solution for the internal elastic force as

$$L = L_{ex} - \sum_{i=1}^{n} l_i \frac{\ddot{q}_i}{\omega_i^2} \tag{10.27}$$

Assuming n to be finite, the solution by the mode-acceleration method, Eq. (10.27), represents an improvement over the results of the mode-displacement method, given by Eq (10.11).

EXAMPLE *10.7*

The solution for the steady-state forced vibration of the uniform rod of Fig. 10-2 is given in Example 10.4. Considering only the fundamental mode, the approximate solution for the longitudinal displacement of the rod is given by

$$\begin{aligned} u(x,t) &= \phi_1(x) q_1(t) \\ &= \left[\frac{16}{\pi^3}\frac{f_0 l^2}{EA} + \frac{4\sqrt{2}}{\pi^2}\cdot\frac{F_0 l}{EA}\right]\frac{1}{1-\dfrac{\omega^2}{\omega_1^2}}\sin\frac{\pi x}{2l}\sin\omega t \end{aligned}$$

In view of Eq. (10.20), the portion of the displacement which results from the inertia forces is given by

$$\begin{aligned} u_{in}(x,t) &= \phi_1(x) q_{1,in}(t) = -\phi_1(x)\frac{\ddot{q}_1(t)}{\omega_1^2} \\ &= \left[\frac{16}{\pi^3}\cdot\frac{f_0 l^2}{EA} + \frac{4\sqrt{2}}{\pi^2}\cdot\frac{F_0 l}{EA}\right]\frac{\dfrac{\omega^2}{\omega_1^2}}{1-\dfrac{\omega^2}{\omega_1^2}}\sin\frac{\pi x}{2l}\sin\omega t \end{aligned}$$

Taking the difference between the expressions given above, we can write the displacement resulting from the external forces as

$$
\begin{aligned}
u_{ex}(x,t) &= \phi_1(x)q_{1,ex}(t) \\
&= \left[\frac{16}{\pi^3}\cdot\frac{f_0l^2}{EA} + \frac{4\sqrt{2}}{\pi^2}\cdot\frac{F_0l}{EA}\right] \sin\frac{\pi x}{2l} \sin \omega t
\end{aligned}
$$

The three expressions given represent the mode-displacement solution for the longitudinal displacement of the rod in terms of the fundamental mode only.

Let us compare the results given above with those of the mode-acceleration method. Applying the external forces statically, we can write the equation of equilibrium of the rod as

$$
\begin{aligned}
EAu'_{ex} &= [f_0(l - x) + F_0] \sin \omega t && \text{for } 0 \le x \le \frac{l}{2} \\
&= f_0(l - x) \sin \omega t && \text{for } \frac{l}{2} \le x \le l
\end{aligned}
$$

Solving this equation with $u(0) = 0$ and with $u(x)$ continuous at $x = l/2$, we can write the exact results for the displacement response to the external forces as

$$
\begin{aligned}
u_{ex} &= \left[\frac{f_0x}{EA}\left(l - \frac{x}{2}\right) + \frac{F_0x}{EA}\right] \sin \omega t && \text{for } 0 \le x \le \frac{l}{2} \\
&= \left[\frac{f_0x}{EA}\left(l - \frac{x}{2}\right) + \frac{1}{2}\cdot\frac{F_0l}{EA}\right] \sin \omega t && \text{for } \frac{l}{2} \le x \le l
\end{aligned}
$$

Then the mode-acceleration solution for the longitudinal displacement of the rod is obtained by adding to this expression for u_{ex} the expression for u_{in} given earlier.

The results of the two methods differ in the expressions for u_{ex}. Considering the response to the distributed and concentrated forces separately, the two solutions for u_{ex} are compared in Fig. 10-6. Evidently there is little advantage in using the mode-acceleration method to obtain the displacement solution. Note that the result given by the mode-displacement method is closer to the exact result for the distributed exciting force than it is for the concentrated exciting force. Generally the displacement shape is smoother for a distributed than for a concentrated exciting force and the approximate series solution more convergent with the mode number.

The approximate solution given by the mode-displacement method for the elastic force in the rod is

$$
\begin{aligned}
L(x,t) &= EA\phi_1'(x)q_1(t) \\
&= \left[\frac{8}{\pi^2}f_0l + \frac{2\sqrt{2}}{\pi}F_0\right] \frac{1}{1 - \dfrac{\omega^2}{\omega_1^2}} \cos\frac{\pi x}{2l} \sin \omega t
\end{aligned}
$$

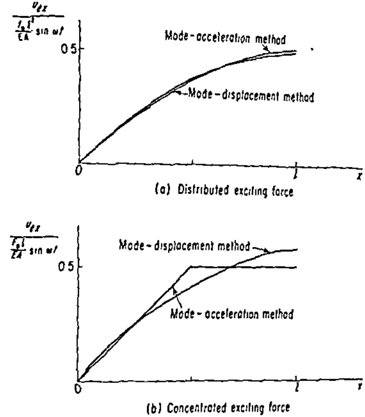


Fig. 10-6 *Comparison of u_{ex} for the mode-displacement and mode-acceleration methods*

We can write the elastic force response to the inertia and external forces, taken separately, as

$$L_{in}(x,t) = EA\phi_1'(x)q_{1,in}(t) = -EA\phi_1'(x)\frac{\ddot{q}_1(t)}{\omega_1^2}$$

$$= \left[\frac{8}{\pi^2}f_0 l + \frac{2\sqrt{2}}{\pi}F_0\right]\frac{\dfrac{\omega^2}{\omega_1^2}}{1-\dfrac{\omega^2}{\omega_1^2}}\cos\frac{\pi x}{2l}\sin\omega t$$

$$L_{ex}(x,t) = EA\phi_1'(x)q_{1,ex}(t)$$

$$= \left[\frac{8}{\pi^2}f_0 l + \frac{2\sqrt{2}}{\pi}F_0\right]\cos\frac{\pi x}{2l}\sin\omega t$$

The exact solution given by the mode acceleration method for the elastic force response to the external force is given by

$$
\begin{aligned}
L_{ex}(x,t) &= EAu'_{ex}(x,t) \\
&= [f_0(l - x) + F_0] \sin \omega t \qquad \text{for} \quad 0 \le x \le \frac{l}{2} \\
&= f_0(l - x) \sin \omega t \qquad \text{for} \quad \frac{l}{2} \le x \le l
\end{aligned}
$$

Considering the responses to the distributed and concentrated exciting forces separately, the two solutions are compared in Fig. 10-7. Obviously there is a greater advantage in using the mode-acceleration method to obtain the elastic force solution than to obtain the displacement solution. Again the result given by the mode-displacement method is closer to the exact result for the distributed exciting force than it is for the concentrated exciting force.

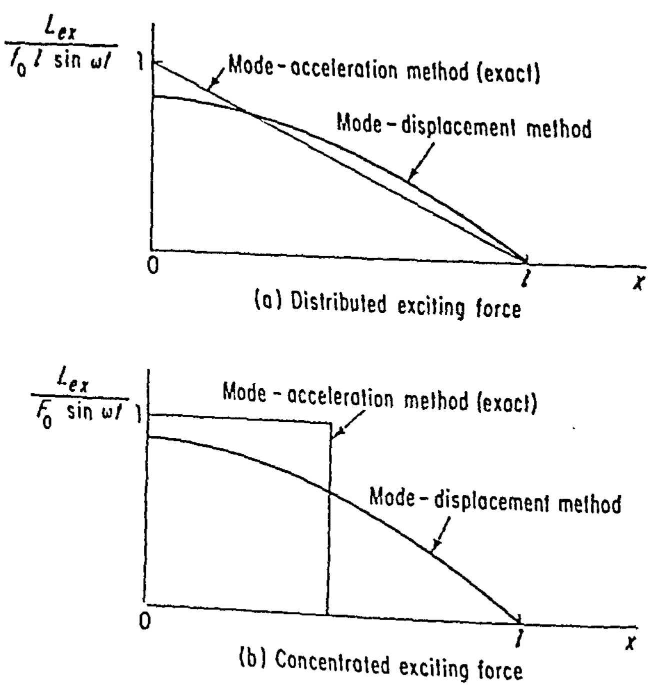


Fig. 10-7 *Comparison of L_{ex} for the mode-displacement and mode-acceleration methods.*

10.4 Systems with Rigid-Body Degrees of Freedom

For a system which has rigid-body degrees of freedom it is convenient to separate the response into two parts, the rigid-body part and the elastic part. The longitudinal displacement of a rod, for example, is given by

$$u(x,t) = u^R(t) + u^E(x,t) \tag{10.28}$$

in which u^R represents the rigid-body translation and u^E the elastic displacement. For this case, the rigid-body displacement is described by the displacement of one normal coordinate q^R, for which $\phi^R(x) = 1$. Using n of the remaining normal coordinates q^E to approximate the elastic displacement, we can write

$$\begin{aligned} u^R &= \phi^R q^R \\ u^E &= \sum_{i=1}^{n} \phi_i^E q_i^E \end{aligned} \tag{10.29}$$

The internal elastic forces in the rod will of course be zero in the rigid-body mode. Letting l_i^E represent the elastic force per unit displacement of the normal coordinate q_i^E, we can write the total elastic force L in the rod as

$$L = \sum_{i=1}^{n} l_i^E q_i^E \tag{10.30}$$

If n is finite, this result will be approximate. The results given by Eqs. (10.28) through (10.30) constitute the mode-displacement method.

Let us consider the mode-acceleration method. The rigid-body displacement is given exactly by the first of Eqs. (10.29). We can separate the elastic displacement into two parts as

$$u^E = u^E_{ex+in_R} + u^E_{in_E} \tag{10.31}$$

in which the first term on the right-hand side represents the elastic displacement resulting from application of the external forces and the rigid-body inertia forces. The rigid-body inertia forces just balance the external forces and result in a condition of dynamic equilibrium. Using the methods of statics and of mechanics of materials, we can obtain the exact solution for the displacement $u^E_{ex+in_R}$. Since the elastic displacement must be orthogonal to the rigid-body displacement, we can write

$$\int_0^l \mu u^E_{ex+in_R}\, dx = 0 \tag{10.32}$$

Thus the elastic displacement is measured relative to a frame of reference attached to the center of mass of the rod. The elastic displacement resulting from application of the elastic motion inertia forces is given by

$$u^E_{in_E} = \sum_{i=1}^{n} \phi_i^E q^E_{i,in}$$

or, using Eq. (10.19), by

$$u^E_{\text{in}_E} = -\sum_{i=1}^{n} \phi_i^E \frac{\ddot{q}_i^E}{\omega_i^2} \tag{10.33}$$

The elastic displacement obtained from Eqs. (10.31) through (10.33) is more complete than that given by the mode-displacement method, the second of Eqs. (10.29). As explained in Sec. 10.3, the difference in the results rests in the completeness with which the response to the external forces is included.

As with the displacement, we can separate the elastic force L in the rod into two parts as

$$L = L_{ex+\text{in}_R} + L_{\text{in}_E} \tag{10.34}$$

Using the methods of statics and of mechanics of materials, we can obtain a direct solution for $L_{ex+\text{in}_R}$. The elastic force resulting from application of the elastic motion inertia forces is given by

$$L_{\text{in}_E} = \sum_{i=1}^{n} l_i^E q_{i,\text{in}}^E$$

or, using Eq. (10.19), by

$$L_{\text{in}_E} = -\sum_{i=1}^{n} l_i^E \frac{\ddot{q}_i^E}{\omega_i^2} \tag{10.35}$$

Thus, the elastic force in the rod according to the mode-acceleration method is given by Eqs. (10.34) and (10.35).

EXAMPLE *10.8*

A uniform rod, free at the ends, is excited by a torque applied at the left-hand end as shown in Fig. 10-8. The natural frequencies and normal mode shapes for the free torsional vibration of the rod are given in Example 8.4. For the rigid body mode

$$\omega^R = 0$$
$$\phi^R(x) = 1$$

and for the elastic modes

$$\omega_i^E = i\pi\sqrt{\frac{GJ}{Il^2}}$$

$$\phi_i^E(x) = \cos\frac{i\pi x}{l} \qquad i = 1, 2, \ldots$$

Assuming that we are interested in the elastic torque distribution in the rod, let us consider the motion in the elastic modes. From Eqs. (8.75), the general-

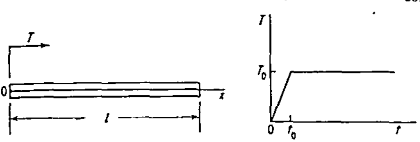


Fig. 10-8

ized masses and stiffnesses associated with the normal coordinates are

$$M_i^E = \int_0^l I\cos^2\frac{i\pi x}{l}\,dx = \frac{Il}{2}$$

$$K_i^E = \int_0^l GJ\left(\frac{i\pi}{l}\right)^2 \sin^2\frac{i\pi x}{l}\,dx = \frac{(i\pi)^2}{2}\frac{GJ}{l} \qquad i = 1, 2, \ldots$$

Considering the work done by the applied torque T in the virtual displacement of a normal coordinate, we can write the generalized external forces as

$$\begin{aligned} Q_i^E &= \frac{T_0}{t_0}t \qquad \text{for} \quad 0 \le t \le t_0 \\ &= T_0 \qquad \text{for} \quad t_0 \le t \qquad i = 1, 2, \ldots \end{aligned}$$

The equations of motion for the normal coordinates are given by

$$\begin{aligned} \frac{Il}{2}\ddot{q}_i^E + \frac{(i\pi)^2}{2}\cdot\frac{GJ}{l}q_i^E &= \frac{T_0}{t_0}t \qquad \text{for} \quad 0 \le t \le t_0 \\ &= T_0 \qquad \text{for} \quad t_0 \le t \qquad i = 1, 2, \ldots \end{aligned}$$

Assuming that the system starts from rest, the solution for the motion of the normal coordinates is given by

$$\begin{aligned} q_i^E &= \frac{2}{(i\pi)^2}\,\frac{T_0 l}{GJ}\left[\frac{t}{t_0} - \frac{1}{\omega_i t_0}\sin\omega_i t\right] \qquad \text{for} \quad 0 \le t \le t_0 \\ &= \frac{2}{(i\pi)^2}\cdot\frac{T_0 l}{GJ}\left\{1 + \frac{1}{\omega_i t_0}\left[\sin\omega_i(t - t_0) - \sin\omega_i t\right]\right\} \\ &\qquad \text{for} \quad t_0 \le t \\ &\qquad i = 1, 2, \ldots \end{aligned}$$

The elastic torque in the rod per unit displacement of the normal coordinates is given by

$$\begin{aligned} T_i^E &= GJ\phi_i'^E \\ &= -i\pi\frac{GJ}{l}\sin\frac{i\pi x}{l} \end{aligned}$$

We can approximate the elastic torque in the rod in terms of the first n normal coordinates by

$$
\begin{aligned}
L &= \sum_{i=1}^{n} l_i^E q_i^E \\
&= -2T_0 \sum_{i=1}^{n} \frac{1}{i\pi} \sin\frac{i\pi x}{l}\left[\frac{t}{t_0} - \frac{1}{\omega_i t_0}\sin\omega_i t\right] \qquad \text{for } 0 \le t \le t_0 \\
&= -2T_0 \sum_{i=1}^{n} \frac{1}{i\pi} \sin\frac{i\pi x}{l}\left\{1 + \frac{1}{\omega_i t_0}[\sin\omega_i(t - t_0) - \sin\omega_i t]\right\} \\
&\qquad\qquad \text{for } \quad t_0 \le t
\end{aligned}
$$

This is the solution given by the mode displacement method. From Eq. (10.35), the elastic torque resulting from application of the elastic motion inertia forces is given by

$$
\begin{aligned}
L_{in_E} &= -\sum_{i=1}^{n} l_i^E \frac{\ddot{q}_i^E}{\omega_i^2} \\
&= 2T_0 \sum_{i=1}^{n} \frac{1}{i\pi} \sin\frac{i\pi x}{l}\cdot\frac{1}{\omega_i t_0}\sin\omega_i t \qquad \text{for } 0 \le t \le t_0 \\
&= 2T_0 \sum_{i=1}^{n} \frac{1}{i\pi} \sin\frac{i\pi x}{l}\cdot\frac{1}{\omega_i t_0}[\sin\omega_i(t - t_0) - \sin\omega_i t] \qquad \text{for } t_0 \le t
\end{aligned}
$$

Taking the difference in the results for L and L_{in_E}, we can write

$$
\begin{aligned}
L_{ex+in_R} &= -2T_0 \sum_{i=1}^{n} \frac{1}{i\pi} \sin\frac{i\pi x}{l}\cdot\frac{t}{t_0} \qquad \text{for } 0 \le t \le t_0 \\
&= -2T_0 \sum_{i=1}^{n} \frac{1}{i\pi} \sin\frac{i\pi x}{l} \qquad \text{for } t_0 \le t
\end{aligned}
$$

This result is approximate as given by the mode-displacement method.

In applying the mode-acceleration method, we will replace the approximate result for L_{ex+in_R} by the exact result. The external torque T and the distributed rigid-body inertia torque $\tau_{in} = -T/l$ are shown in Fig. 10-9.

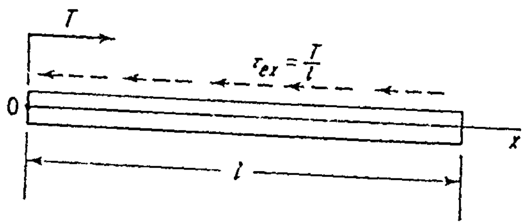


Fig. 10-9 *External and rigid-body inertia forces on the system of Fig. 10-8.*

Then the exact solution for the elastic torque resulting from the external torque and the rigid-body inertia torques is given by

$$
\begin{aligned}
L_{ex+in_R} &= \left(1 - \frac{x}{l}\right) \frac{T_0}{t_0} t \qquad \text{for } 0 \le t \le t_0 \\
&= \left(1 - \frac{x}{l}\right) T_0 \qquad \text{for } t_0 \le t
\end{aligned}
$$

The elastic torque in the rod given by the mode-acceleration method follows from Eq. (10.34).

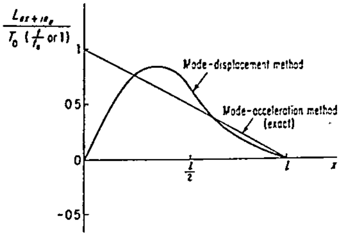


Fig. 10-10 *Comparison of L_{ex+in_R} obtained by the mode-displacement method and the mode-acceleration method.*

Let us compare the approximate and the exact results for L_{ex+in_R}. Along with the exact solution, Fig. 10-10 shows the approximate solution containing the contributions of the first two normal modes. The difference is quite large near the left-hand end of the rod. As more terms are added to the approximate solution, the result will of course approach the exact result.

It is instructive to compare the solutions for L_{in_E} and L_{ex+in_R} given by the mode-displacement method. In the higher modes of vibration, for which $\omega_i t_0$ is a large number, the terms in L_{in_E} are much smaller than the terms in L_{ex+in_R}. Thus the series solution for L_{in_E} is much more convergent. In the mode-acceleration method, we replace the approximate series solution for L_{ex+in_R}, the least convergent series, with the exact solution. Evidently the solution given by the mode-acceleration method is much more convergent than that given by the mode-displacement method.

10.5 Displacement Excitation

So far we have considered that the exciting forces are given. On some occasions, we will know the motion of the points of excitation rather than the magnitude of the forces. For a prescribed motion of one or more points of a system, it will be convenient to separate the displacement response into two parts as was done in Sec. 10.3. As an example, we can write the longitudinal displacement of a rod as

$$u(x,t) = u_{ex}(x,t) + u_{in}(x,t) \tag{10.36}$$

in which u_{ex} represents the displacement resulting from the prescribed displacements of the points of excitation, not considering inertia forces. The displacement u_{in} is that which results from the inertia forces that act on the rod.

Suppose the rod is given a displacement $u_1(t)$ at the point $x = x_1$. Using the methods of statics and of mechanics of materials, we can determine the displacement of the rod $\gamma_1(x)$ resulting from a unit static displacement of the point $x = x_1$. Then we can write that

$$u_{ex}(x,t) = \gamma_1(x)u_1(t) \tag{10.37}$$

Since $\gamma_1(x_1) = 1$, the displacement u_{ex} at the point $x = x_1$ matches the prescribed displacement. Evidently the displacement u_{in} must be zero at the point $x = x_1$. We can approximate u_{in} in terms of n normal coordinates $q_{i,in}$ by

$$u_{in}(x,t) = \sum_{i=1}^{n} \phi_i(x)q_{i,in}(t) \tag{10.38}$$

in which the normal mode shapes must satisfy the condition that $\phi_i(x_1) = 0$. Thus the normal mode shapes are those for the system with the point $x = x_1$ fixed.

Let us write the equations of motion for the normal coordinates. From Eq. (8.65), the generalized inertia force associated with the coordinate $q_{i,in}$ resulting from the motion u_{in} is given by $-M_i\ddot{q}_{i,in}$. The rod experiences the additional distributed inertia force $-\mu\ddot{u}_{ex} = -\mu\gamma_1\ddot{u}_1$. In a virtual displacement of the ith normal coordinate, the work done by the additional inertia force is

$$\delta W = -\delta q_i\ddot{u}_1 \int_0^l \mu\gamma_1\phi_i\, dx$$

Thus the additional generalized inertia force is given by $-\ddot{u}_1 \int_0^l \mu\gamma_1\phi_i\, dx$. We can write the total generalized inertia force associated with $q_{i,in}$ as

$$Q_{i,in} = -M_i\ddot{q}_{i,in} - \ddot{u}_1 \int_0^l \mu\gamma_1\phi_i\, dx \tag{10.39}$$

From Eq. (8.69), the generalized elastic force associated with $q_{i,in}$ is

$$Q_{i,el} = -K_i q_{i,in} \tag{10 40}$$

Since the normal mode shapes have zero amplitude at the point $x = x_1$, the unknown exciting force at that point will not excite the normal coordinates. We can write the equations for the normal coordinates as

$$\sum Q_i = -M_i \ddot{q}_{i,in} - K_i q_{i,in} - \ddot{u}_1 \int_0^l \mu \gamma_1 \phi_i \, dx = 0 \tag{10 41}$$

Given the prescribed motion $u_1(t)$, we can solve Eqs. (10.41) using the methods of Chap. 3.

We can write the internal elastic force in the rod as

$$L(x,t) = L_{ex}(x,t) + L_{in}(x,t) \tag{10 42}$$

in which L_{ex} represents the elastic force resulting from the prescribed displacement of the point $x = x_1$. Letting l_i represent the elastic force per unit displacement of $q_{i,in}$

$$L_{in}(x,t) = \sum_{i=1}^{n} l_i(x) q_{i,in}(t) \tag{10 43}$$

EXAMPLE *10.9*

A uniform string of mass per unit length μ is stretched with a large tensile force T. As shown in Fig. 10-11, the string is excited by a prescribed harmonic displacement of the midpoint. We expect the response to be symmetric around

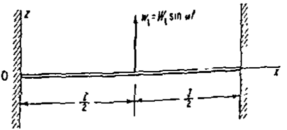


Fig. 10-11

the midpoint. Thus we need only consider the response of one-half of the string. Neglecting the inertia forces, we can write the displacement of the first half of the string as

$$\begin{aligned} w_{ex} &= \gamma_1(x) w_1(t) \\ &= \frac{2k}{l} W_1 \sin \omega t \qquad \text{for } 0 \le x \le \frac{l}{2} \end{aligned}$$

The normal mode shapes and natural frequencies of the first half of the string with the midpoint fixed are given by

$$\phi_i(x) = \sin\frac{2i\pi x}{l}$$
$$\omega_i = 2i\pi\sqrt{\frac{T}{\mu l^2}} \qquad i = 1, 2, \ldots$$

We can write the generalized masses for the first half of the string as

$$M_i = \int_0^{l/2} \mu\phi_i^2\,dx = \mu\int_0^{l/2} \sin^2\frac{2i\pi x}{l}\,dx$$
$$= \frac{\mu l}{4}$$

and the generalized stiffnesses as

$$K_i = M_i\omega_i^2 = \frac{\mu l}{4}\omega_i^2$$

The first half of the string experiences a distributed transverse inertia force $-\mu\ddot{w}_{ex} = \mu\omega^2\frac{2x}{l}W_1\sin\omega t$. In a virtual displacement of the ith normal coordinate, the work done by this inertia force is given by

$$\delta W = -\int_0^{l/2} \mu\ddot{w}_{ex}\phi_i\,dx\,\delta q_{i,in}$$
$$= \frac{2\mu\omega^2}{l}W_1\sin\omega t\int_0^{l/2} x\sin\frac{2i\pi x}{l}\,dx\,\delta q_{i,in}$$
$$= \frac{(-1)^{i+1}}{2i\pi}\mu l\omega^2 W_1\sin\omega t\,\delta q_{i,in}$$

We can write the generalized inertia and elastic forces as

$$Q_{i,in} = -\frac{\mu l}{4}\ddot{q}_{i,in} + \frac{(-1)^{i+1}}{2i\pi}\mu l\omega^2 W_1\sin\omega t$$
$$Q_{i,el} = -\frac{\mu l}{4}\omega_i^2 q_{i,in} \qquad i = 1, 2, \ldots$$

and the equations of motion as

$$\frac{\mu l}{4}\ddot{q}_{i,in} + \frac{\mu l}{4}\omega_i^2 q_{i,in} = \frac{(-1)^{i+1}}{2i\pi}\mu l\omega^2 W_1\sin\omega t$$

If we are interested only in the steady-state forced vibration, we can write the solution for the motion of the normal coordinates as

$$q_{i,in} = \frac{2(-1)^{i+1}}{i\pi}\cdot\frac{\dfrac{\omega^2}{\omega_i^2}}{1-\dfrac{\omega^2}{\omega_i^2}}W_1\sin\omega t \qquad i = 1, 2, \ldots$$

Then the motion of the first half of the string is approximated by

$$w = w_{ex} + \sum_{i=1}^{n} \phi_i q_{i,1n}$$

$$= \left[\frac{x}{l} + \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i\pi} \frac{\frac{\omega^2}{\omega_i^2}}{1 - \frac{\omega^2}{\omega_i^2}} \sin \frac{2i\pi x}{l} \right] 2W_1 \sin \omega t$$

Considering the forces acting on an element of the string at the midpoint, we can write the exciting force required as

$$F_1 = 2Tw'\left(\frac{l}{2}\right)$$

$$= 4T \frac{W_1}{l} \left[1 - 2 \sum_{i=1}^{n} \frac{\frac{\omega^2}{\omega_i^2}}{1 - \frac{\omega^2}{\omega_i^2}} \right] \sin \omega t$$

EXAMPLE *10 10*

A uniform beam pinned at one end is excited by a prescribed transverse displacement of the other end as shown in Fig. 10-12. Assuming that no inertia forces are acting, we can write the displacement of the beam as

$$w_{ex} = \frac{x}{l} w_1(t)$$

From Example 8 5 the normal mode shapes and natural frequencies for a uniform simply supported beam are

$$\phi_i(x) = \sin \frac{i\pi x}{l}$$

$$\omega_i = (i\pi)^2 \sqrt{\frac{EI}{\mu l^4}} \qquad i = 1, 2, \ldots$$

From Eqs. (8.82) we can write the generalized masses and stiffnesses as

$$M_i = \int_0^l \mu \sin^2 \frac{i\pi x}{l}\, dx = \frac{\mu l}{2}$$

$$K_i = \int_0^l EA\left(\frac{i\pi}{l}\right)^2 \cos^2 \frac{i\pi x}{l}\, dx = \frac{(i\pi)^2}{2} \frac{EA}{l} \qquad i = 1, 2, \ldots$$

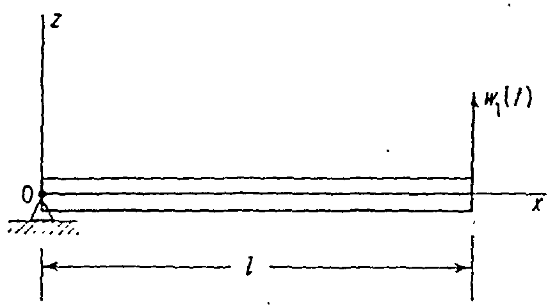


Fig. 10-12

The distributed transverse inertia force resulting from the motion w_{ex} is given by $-\mu \frac{x}{l} \ddot{w}_1$. Then the work done by this force in a virtual displacement of the ith normal coordinate is

$$\begin{aligned}\delta W &= -\int_0^l \mu \frac{x}{l} \ddot{w}_1 \sin \frac{i\pi x}{l}\, dx\, \delta q_{i,in} \\ &= \frac{(-1)^i}{i\pi} \mu l \ddot{w}_1\, \delta q_{i,in}\end{aligned}$$

The generalized inertia and elastic forces are given by

$$\begin{aligned}Q_{i,in} &= -\frac{\mu l}{2} \ddot{q}_{i,in} + \frac{(-1)^i}{i\pi} \mu l \ddot{w}_1 \\ Q_{i,el} &= -\frac{(i\pi)^2}{2} \cdot \frac{EA}{l} q_{i,in}\end{aligned}$$

and the equations of motion by

$$\frac{\mu l}{2} \ddot{q}_{i,in} + \frac{(i\pi)^2}{2} \cdot \frac{EA}{l} q_{i,in} = \frac{(-1)^i}{i\pi} \mu l \ddot{w}_1 \qquad i = 1, 2, \ldots$$

Given the nature of the motion w_1, we can obtain the response of the normal coordinates and the solution for the motion of the beam given by

$$w = w_{ex} + \sum_{i=1}^{n} \phi_i q_{i,in}$$

Neglecting the inertia forces, the bending moment in the beam is given by $L_{ex} = 0$. The bending moment in the beam resulting from a unit displacement of the ith normal coordinate is given by

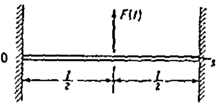
$$\begin{aligned}l_i &= EI\phi_i'' \\ &= (i\pi)^2 \frac{EI}{l^2}\end{aligned}$$

We can write the solution for the bending moment in the beam as

$$L = L_{tn} = \sum_{i=1}^{n} l q_{i,tn}$$

Problems

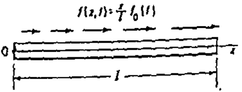
10-1 A concentrated transverse force $F(t)$ acts on a uniform string of mass per unit length μ as shown. The tension T in the string is considered to be large. For forced transverse vibrations of small amplitude, write the equations of motion in the normal coordinates. The normal mode shapes and frequencies are given by Prob. 8-8.



Prob. 10-1

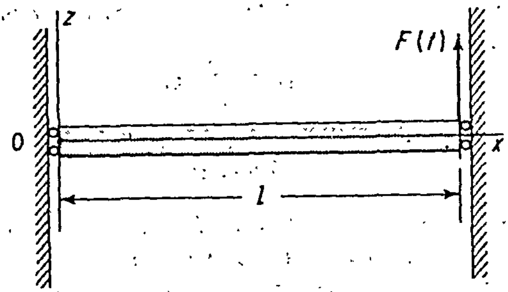
10-2 Forced vibrations are excited in a uniform rod built in at both ends by a uniformly distributed torque $\tau(t)$ per unit length. Give the equations for the torsional motion in the normal coordinates.

10-3 A uniform rod, free at both ends, is excited by a distributed longitudinal force per unit length given by $f(x,t) = \frac{x}{l} f_0(t)$. Obtain the equations of forced motion in the normal coordinates. Refer to Prob. 8-6 for the natural frequencies and normal mode shapes.



Prob. 10-3

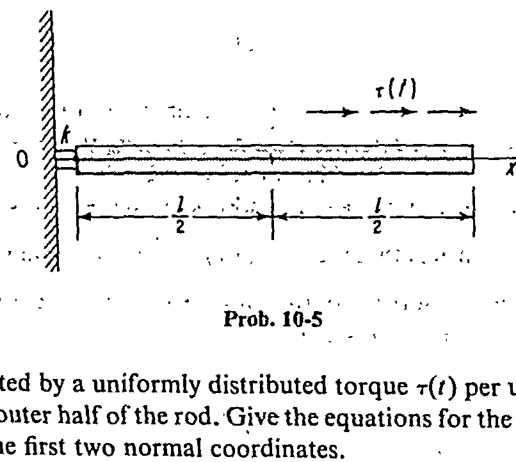
10-4 The ends of a uniform beam are free to translate in the transverse direction but are constrained against rotation as shown. A concentrated exciting force $F(t)$ acts at the right-hand end of the beam. Write the equations



Prob. 10-4

of motion for the beam in the normal coordinates. The natural frequencies and normal mode shapes are given by Prob. 8-11.

10-5 A uniform rod is connected to a support through a torsional spring of stiffness k as shown. For the special case in which $\frac{kl}{GJ} = 1$, the first two natural frequencies and normal mode shapes are given by Example 8.9. The



Prob. 10-5

system is excited by a uniformly distributed torque $\tau(t)$ per unit length which acts over the outer half of the rod. Give the equations for the torsional motion in terms of the first two normal coordinates.

10-6 For the system of Fig. 8-16, the mass μl of the uniform rod is equal to the suspended mass m. An approximate result is given in Example 9.8 for the fundamental natural frequency and normal mode shape. Write the equation of motion in the fundamental normal coordinate if a concentrated longitudinal force $F(t)$ is applied to the mass.

10-7 The approximate fundamental mode shape for the flexural vibration of the uniform beam shown is given in terms of the coordinates w_1, w_2, w_3 by Prob. 9-18. A distributed transverse force $f(x,t) = \frac{x}{l} f_0(t)$ acts on the

Write the steady-state solution for the bending moment in the fundamental mode.

10-16 For the uniform rod of Probs. 10-2 and 10-10, compare the solutions for the elastic torque given by the mode-displacement and mode-acceleration methods. Consider only the fundamental mode.

10-17 Compare the mode-displacement and mode-acceleration solutions for the elastic torque in the uniform rod of Probs. 10-5 and 10-12. Give the solutions in terms of the first two normal modes.

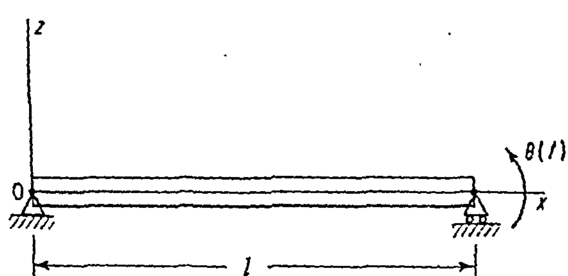
10-18 For the system of Probs. 10-6 and 10-13, compare the solutions for the elastic force in the rod given by the mode-displacement and mode-acceleration methods. Consider only the fundamental mode.

10-19 Compare the mode-displacement and mode-acceleration solutions for the displacement response of the beam of Probs. 10-7 and 10-14. Give the solutions in terms of the fundamental mode. Use the trapezoidal rule in the numerical integration.

10-20 For the free-free rod of Probs. 10-3 and 10-11, compare the results for the elastic force in the rod given by the mode-displacement and mode-acceleration methods. Consider only the first elastic mode.

10-21 Compare the solutions given by the mode-displacement and mode-acceleration methods for the elastic bending moment in the beam of Probs. 10-8 and 10-15. Give the solutions in terms of the first elastic mode.

10-22 The right-hand end of a uniform rod fixed at the left-hand end is given the torsional motion $\theta(t) = \theta_0 \sin \omega t$. Obtain the steady-state solution for the elastic torque in the rod.



Prob. 10-23

10-23 The right-hand end of a uniform simply supported beam is given an angular motion described by $\theta(t) = \theta_0\left(1 - \cos \dfrac{2\pi t}{t_0}\right)$ for $0 \le t \le t_0$ and $\theta(t) = 0$ for $t \ge t_0$. Determine the displacement response of the beam.

Matrices

A.1 Definitions.

A matrix is a rectangular array which satisfies certain operational rules. For example, the matrix

$$[a] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \vdots \\ \vdots & & & \vdots \\ a_{m1} & \cdot & \cdots & a_{mn} \end{bmatrix}$$

has m rows and n columns and is said to be of order $m \times n$

A matrix having a single column is called a column matrix, symbolized by { }. Similarly a matrix having one row is identified by ⌊ ⌋.

If the number of rows m equals the number of columns n we have a square matrix. We can associate a determinant with a square matrix. A square matrix in which the elements a_{ii} on the diagonal are the only nonzero elements is called a diagonal matrix, symbolized by [\] For example,

$$[a] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

A unit matrix is a diagonal matrix whose nonzero elements are all equal to unity, identified by $[I]$. For example,

$$[I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The transpose of a matrix, symbolized by [ ]', is formed by interchanging the rows and columns A square matrix which is equal to its transpose is called a symmetric matrix. For example, the matrix

$$[a] = [a]' = \begin{bmatrix} 3 & -1 & 4 \\ -1 & 2 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

is symmetric.

A.2 Rules of operation.

Two matrices are equal if their corresponding elements are equal. The matrices must be of the same order $m \times n$.

If two matrices are of the same order, the operations of addition or subtraction may be performed. This involves the addition or subtraction of corresponding elements. For example,

$$\begin{bmatrix} 1 & -3 & 2 \\ 4 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -6 & 3 \\ 4 & 5 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -9 & 5 \\ 8 & 7 & 1 \end{bmatrix}$$

Multiplication of a matrix by a number is accomplished by multiplying each of the elements by the number. For example,

$$3\begin{bmatrix} 2 & -3 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -9 \\ 15 & 12 \end{bmatrix}$$

Evidently a number can be factored out of a matrix by factoring it out of each element.

Two matrices $[a]$ and $[b]$ can be multiplied, as expressed by

$$[a][b] = [c]$$

provided that they are conformable. For the order of multiplication indicated, the matrices are conformable if the number of columns of $[a]$ is equal to the number of rows of $[b]$. Consider the matrix product

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

As an example, the element c_{21} is given by

$$c_{21} = a_{21}b_{11} + a_{22}b_{21}$$

which follows from multiplying the elements of the second row of $[a]$ by the corresponding elements of the first column of $[b]$ and adding the results. For the given matrices, the matrix product $[b][a]$ cannot be defined since the matrices are not conformable in that order. In general, matrix multiplication is not commutative and $[a][b] \neq [b][a]$. Following are some examples of matrix multiplication.

$$\lfloor 2 \;\; -1 \;\; 3 \rfloor \begin{Bmatrix} 4 \\ 1 \\ 5 \end{Bmatrix} = 2 \times 4 + (-1) \times 1 + 3 \times$$

$$= 22$$

$$\begin{bmatrix} 4 & -1 \\ 3 & 6 \end{bmatrix} \begin{Bmatrix} 3 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 11 \\ 15 \end{Bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 3 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 19 \\ 14 & 2 & 26 \\ 13 & -11 & 10 \end{bmatrix}$$

In matrix algebra, the operation analogous to division involves multiplication by the inverse matrix. The inverse of $[a]$, identified by $[a]^{-1}$, is defined by

$$[a][a]^{-1} = [I]$$

where $[I]$ is the unit matrix. The inverse matrix can be determined by the following procedure. First, let us determine the minor M_{ij} of each element of the matrix $[a]$. Then the cofactors A_{ij} are given by

$$A_{ij} = (-1)^{i+j} M_{ij}$$

The matrix of the cofactors $[A]$ is often called the adjoint matrix. The inverse matrix is given by the transpose of the adjoint matrix divided by the determinant $|a|$, as follows

$$[a]^{-1} = \frac{[A]'}{|a|}$$

Evidently $[a]$ must be a square matrix. It is necessary that $|a| \neq 0$. Consider the following numerical example:

$$[a] = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \quad |a| = 12 - 2 = 10$$

$$[A] = \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$$

$$[a]^{-1} = \frac{\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}}{10} = \begin{bmatrix} 0.4 & -0.2 \\ -0.1 & 0.3 \end{bmatrix}$$

$$[a][a]^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

It can be shown that the transpose of a matrix product is given by the product in the reverse order of the transposes of the individual matrices. Thus

$$[[a][b]]' = [b]'[a]'$$

Similarly, for the inverse of a matrix product

$$[[a][b]]^{-1} = [b]^{-1}[a]^{-1}$$

A.3 Representation of linear equations

We can write a set of simultaneous algebraic equations in matrix form. For example,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= y_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= y_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= y_3 \end{aligned}$$

is represented by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}$$

or by

$$[a]\{x\} = \{y\}$$

The equivalence is established by the rules for multiplication and for equality. We can solve the equations for $\{x\}$ by premultiplying the equation by the inverse of $[a]$, leading to

$$[a]^{-1}[a]\{x\} = [a]^{-1}\{y\}$$
$$\{x\} = [a]^{-1}\{y\}$$

The Generalized Inertia Force

Let us consider an n-degree-of-freedom system whose motion is described in terms of the n generalized coordinates $q_1, \ldots, q_n$. Then the position of the ith particle of mass of the system with respect to an inertial observer is given in general by

$$\bar{r}_i = \bar{r}_i(q_1, \ldots, q_n, t) \tag{B.1}$$

We can write the velocity of the particle as

$$\dot{\bar{r}}_i = \sum_{j=1}^{n} \frac{\partial \bar{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \bar{r}_i}{\partial t} \tag{B.2}$$

From Eq. (B.2)

$$\frac{\partial \dot{\bar{r}}_i}{\partial \dot{q}_j} = \frac{\partial \bar{r}_i}{\partial q_j} \tag{B.3}$$

With the system in motion, the inertia force on the ith particle of mass m_i is represented by $-m_i\ddot{\bar{r}}_i$. In a virtual displacement of the system, the ith particle is displaced by $\delta\bar{r}_i$. Then the virtual work done by the inertia force on the particle is given by $(-m_i\ddot{\bar{r}}_i) \cdot \delta\bar{r}_i$. For the system

$$\delta W_{in} = \sum_i (-m_i\ddot{\bar{r}}_i) \cdot \delta\bar{r}_i \tag{B.4}$$

From Eq. (B.1), the virtual displacements of the particles are related to those of the generalized coordinates by

$$\delta\bar{r}_i = \sum_{j=1}^{n} \frac{\partial \bar{r}_i}{\partial q_j} \delta q_j \tag{B.5}$$

In view of Eqs. (B.4) and (B.5), we can write

$$\delta W_{in} = \sum_i \sum_{j=1}^{n} (-m_i\ddot{\bar{r}}_i) \cdot \frac{\partial \bar{r}_i}{\partial q_j} \delta q_j \tag{B.6}$$

We can define the generalized inertia forces associated with the generalized coordinates by

$$\delta W_{in} = \sum_{j=1}^{n} Q_{j\,in} \delta q_j \tag{B.7}$$

Comparing Eqs. (B.6) and (B.7), the generalized inertia forces are

$$Q_{j,\text{in}} = \sum_i (-m_i\ddot{\bar{r}}_i)\cdot\frac{\partial\bar{r}_i}{\partial q_j} \tag{B.8}$$

This result can be rewritten as

$$Q_{j,\text{in}} = -\sum_i \left[\frac{d}{dt}\left(m_i\dot{\bar{r}}_i\cdot\frac{\partial\bar{r}_i}{\partial q_j}\right) - m_i\dot{\bar{r}}_i\cdot\frac{d}{dt}\left(\frac{\partial\bar{r}_i}{\partial q_j}\right)\right] \tag{B.9}$$

Let us assume that we can interchange the order of differentiation in the term

$$\frac{d}{dt}\left(\frac{\partial\bar{r}_i}{\partial q_j}\right) = \frac{\partial\dot{\bar{r}}_i}{\partial q_j}$$

With this result and that of Eq. (B.3), we can write Eq. (B.9) as

$$Q_{j,\text{in}} = -\sum_i \left[\frac{d}{dt}\left(m_i\dot{\bar{r}}_i\cdot\frac{\partial\dot{\bar{r}}_i}{\partial\dot{q}_j}\right) - m_i\dot{\bar{r}}_i\cdot\frac{\partial\dot{\bar{r}}_i}{\partial q_j}\right] \tag{B.10}$$

or as

$$Q_{j,\text{in}} = -\sum_i \left\{\frac{d}{dt}\left[\frac{\partial}{\partial\dot{q}_j}\left(\tfrac{1}{2}m_i\dot{\bar{r}}_i\cdot\dot{\bar{r}}_i\right)\right] - \frac{\partial}{\partial q_j}\left(\tfrac{1}{2}m\dot{\bar{r}}_i\cdot\dot{\bar{r}}_i\right)\right\} \tag{B.11}$$

Interchanging the order of summation and differentiation

APPENDIX C

Characteristics of the Flexural Normal Modes of Uniform Beams

The natural frequencies and normal mode shapes of a uniform beam result from the introduction of the appropriate end conditions from Eqs (8.54) into the general solution for the shape, Eq (8.56) Letting k represent the quantity $\sqrt[4]{\frac{\mu\omega^2}{EI}}$ from Eq (8 56), we can write the natural frequency as

$$\omega = (kl)^2\sqrt{\frac{EI}{\mu l^4}}$$

Thus the dimensionless quantity kl is a measure of the natural frequency In the following, the values for kl are given for the first three normal modes for several common combinations of end conditions In addition the normal mode shapes are given. Note in particular the asymptotic values of the given quantities

SIMPLY SUPPORTED BEAM (PINNED-PINNED)

End conditions

$$W(0) = W''(0) = W(l) = W''(l) = 0$$

Normal mode shape of nth mode

$$\phi_n(x) = \sin\frac{n\pi x}{l}$$

Values of kl given by

$$kl = n\pi$$

FREE-FREE BEAM

End conditions

$$W''(0) = W'''(0) = W''(l) = W'''(l) = 0$$

Normal mode shape of nth mode

$$\phi_n(x) = \cosh k_n x + \cos k_n x - B_n(\sinh k_n x + \sin k_n x)$$

Comparing Eqs. (B.6) and (B.7), the generalized inertia forces are

$$Q_{j,\text{in}} = \sum_i (-m_i \ddot{\bar{r}}_i) \cdot \frac{\partial \bar{r}_i}{\partial q_j} \tag{B.8}$$

This result can be rewritten as

$$Q_{j,\text{in}} = -\sum_i \left[\frac{d}{dt}\left(m_i \dot{\bar{r}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}\right) - m_i \dot{\bar{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right)\right] \tag{B.9}$$

Let us assume that we can interchange the order of differentiation in the term

$$\frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right) = \frac{\partial \dot{\bar{r}}_i}{\partial q_j}$$

With this result and that of Eq. (B.3), we can write Eq. (B.9) as

$$Q_{j,\text{in}} = -\sum_i \left[\frac{d}{dt}\left(m_i \dot{\bar{r}}_i \cdot \frac{\partial \dot{\bar{r}}_i}{\partial \dot{q}_j}\right) - m_i \dot{\bar{r}}_i \cdot \frac{\partial \dot{\bar{r}}_i}{\partial q_j}\right] \tag{B.10}$$

or as

$$Q_{j,\text{in}} = -\sum_i \left\{\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\tfrac{1}{2} m_i \dot{\bar{r}}_i \cdot \dot{\bar{r}}_i\right)\right] - \frac{\partial}{\partial q_j}\left(\tfrac{1}{2} m \dot{\bar{r}}_i \cdot \dot{\bar{r}}_i\right)\right\} \tag{B.11}$$

Interchanging the order of summation and differentiation

$$Q_{j,\text{in}} = -\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \tfrac{1}{2} m_i \dot{\bar{r}}_i \cdot \dot{\bar{r}}_i\right)\right] + \frac{\partial}{\partial q_j}\left(\sum_i \tfrac{1}{2} m_i \dot{\bar{r}}_i \cdot \dot{\bar{r}}_i\right) \tag{B.12}$$

Note that the kinetic energy of the system is given by

$$T = \sum_i \frac{1}{2} m_i \dot{\bar{r}}_i \cdot \dot{\bar{r}}_i$$

Thus, Eq. (B.12) becomes

$$Q_{j,\text{in}} = -\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial T}{\partial q_j} \tag{B.13}$$

This result permits us to determine the generalized inertia forces from the expression for the kinetic energy of the system.

APPENDIX C

Characteristics of the Flexural Normal Modes of Uniform Beams

The natural frequencies and normal mode shapes of a uniform beam result from the introduction of the appropriate end conditions from Eqs. (8 54) into the general solution for the shape, Eq. (8 56) Letting k represent the quantity $\sqrt[4]{\frac{\mu\omega^2}{EI}}$ from Eq (8.56), we can write the natural frequency as

$$\omega = (kl)^2\sqrt{\frac{EI}{\mu l^4}}$$

Thus the dimensionless quantity kl is a measure of the natural frequency. In the following, the values for kl are given for the first three normal modes for several common combinations of end conditions In addition the normal mode shapes are given. Note in particular the asymptotic values of the given quantities.

Simply Supported Beam (Pinned-Pinned)

End conditions

$$W(0) = W''(0) = W(l) = W''(l) = 0$$

Normal mode shape of nth mode

$$\phi_n(x) = \sin\frac{n\pi x}{l}$$

Values of kl given by

$$kl = n\pi$$

Free-Free Beam

End conditions

$$W''(0) = W'''(0) = W''(l) = W'''(l) = 0$$

Normal mode shape of nth mode

$$\phi_n(x) = \cosh k_n x + \cos k_n x - B_n(\sinh k_n x + \sin k_n x)$$

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The final three chapters are devoted to continuous systems. In keeping with the improved background of modern students in mathematics and computer usage, Professor Anderson has placed more emphasis than is usual in other books on the subject of multiple degree of freedom and continuous systems. He has also given special emphasis to the forced vibrations of such systems.

In keeping with the modernity of the text, the author uses a multi-viewed approach in attacking problems. In two degree of freedom problems, for example, the author sets up equations of motion by summing forces, from the principles of virtual work, and by stiffness and flexibility influence coefficients. The author gives approximate methods for obtaining normal modes based on both the work and energy approaches and on the equations of motion. He also provides approximate solutions for the motion or force response, using the mode displacement or mode acceleration methods. Displacement excitation is given full coverage.

Both the text material and the problems are profusely illustrated. In addition to approximately 200 class-tested problems, graded in level of difficulty, many illustrative examples are given. Appendices include valuable material on matrices, the generalized inertia force, and characteristics of the flexural normal modes of uniform beams.